

PLASTICITY OF CRYSTALS

WITH SPECIAL REFERENCE TO METALS

BY

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AND

DR.-ING. W. BOAS

A translation from the German by F. A. Hughes & Co., Limited,
of
"KRISTALLPLASTIZITAET
mit besonderer Berücksichtigung der Metalle."

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TRANSLATORS' PREFACE

When Schmid and Boas decided to publish in book form the results of their widespread investigations into the mechanism of the deformation of metals, they rendered a great service to all those in science and industry interested in studying the same problem.

This book, with its lucid exposition and wide range, is cited as the first reference in innumerable metallurgical papers, and became a classic within a year or two of its publication.

The major problems resulting from the complex deformation behaviour of magnesium led the scientists concerned with that metal to welcome the concentrated knowledge on the subject contained in Schmid and Boas' publication.

In accordance with our resolve to make available to the magnesium and other industries new publications of value to their research and development, we decided to add the translation of this book to our previous publications of Beck and of Bulian and Fahrenhorst.

Since the appearance of Schmid and Boas' book there has been a great development in the field treated, and many aspects of strength and plasticity have changed considerably. However, no text-book has appeared that could replace Schmid-Boas in every respect, and it has remained as indispensable to the research worker as it was when first published. Information about the most important changes that have taken place in the meantime can be obtained from a less detailed treatment of the subject in the book *An Introduction to the Physics of Metals and Alloys*, by Dr. W. Boas (Melbourne University Press), while the structural aspects of plastic deformation are dealt with in C. S. Barrett's book, *Structure of Metals* (McGraw-Hill Book Company, Inc.).

The translation of *Kristallplastizitaet* has taken a long time, but we believe its contents to be of great value, and it is our hope that the provision of an English text will materially assist all those researchers who are interested in the deformation and plasticity of crystals.

In conclusion we wish to express our thanks to Dr. W. H. Taylor, Dr. E. Orowan and Mr. R. W. K. Honeycombe of the Cavendish Laboratory, Cambridge, for revising the translation and aiding us to prepare it for the press, and to Mr. L. H. Tripp (the translator), who has carried the main burden.

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January 1950.

When the German edition was published Dr. E. Schmid was Professor of Physics at Fribourg University (Switzerland) and foreign correspondent of the Kaiser Wilhelm-Gesellschaft ; he is at present with the Metallgesellschaft A.G., Frankfort-on-Main. Dr. W. Boas was at the same university, and is now Chief of Division of Tribophysics, Commonwealth Scientific and Industrial Research Organization, Melbourne.

FOREWORD

Plasticity is that property of solids by virtue of which they change their shape permanently under the influence of external forces. Although this property has been exploited since the earliest days of human history—an exploitation which, thanks to modern technical methods, has now reached a very high level of perfection—and although unceasing efforts have been made to obtain a clear picture and a scientific explanation of the processes involved in the phenomenon of deformation, so far neither a full description nor an entirely adequate theoretical interpretation has been possible.

In the present work, which is based on lectures which I delivered in 1930–31 at the Technical High School in Berlin, we describe what is known about the plastic behaviour of a specially important class of solids—crystals. Since the deformation of crystal aggregates is mainly governed by the deformation of the individual grain, the latter provides the foundation for our knowledge of the plasticity of crystalline materials in general. In the last twenty years we have learnt a great deal about crystal plasticity, in the first place, owing to the development of methods for growing crystals, which have enormously increased the experimental material available, and secondly, as a result of the application of X-ray diffraction methods to the investigation of solids.

This book is addressed to a large circle of readers. The experimental data which it presents in classified form, and which it attempts to interpret, should assist the physicist to evolve a theory of plasticity. It brings to the notice of the crystallographer and mineralogist those researches into metal crystals which have for their particular object the dynamics of crystal deformation. The geologist will discover, in the development of textures in cast and wrought metals, analogies with similar phenomena in his own field of enquiry, and he should therefore find our tentative explanations instructive. Workers in the field of metals research and technology will find in this book the crystallographic and physical principles underlying the plastic behaviour of their material, and they will be shown by examples how our knowledge of the polycrystalline state can be both increased and applied. Technologist and designer will become familiar with that mass of data from which the technological characteristics of metals are derived. In this way the fundamental significance of the constants employed, and, in particular, the

possibility of changing them during operations, will be made clear. Last but not least, it is hoped that all those who themselves are studying the plasticity of crystalline materials will be helped by this book in their choice of experimental technique and methods of research.

I wish to express my gratitude to Mr. M. Polanyi, who introduced me to this subject many years ago, who has since been of great assistance to me, and to whose inspiration this book is due. Sincere thanks are also extended to all my collaborators during the happy years of work at the Kaiser Wilhelm Institute for Fibre Chemistry and Metal Research in Berlin, and especially to Messrs. S. Wassermann, W. Boas (co-author of this book), W. Fahrenhorst and G. Siebel (Bitterfeld). I am also grateful to the Notgemeinschaft der Deutschen Wissenschaft for their continued assistance.

I should like to thank numerous colleagues for permission to reproduce illustrations and diagrams from their works; and I am also indebted to the publishers for their co-operation, and for the very helpful way in which they have met my wishes.

Fribourg, Switzerland.
January 1935.

ERICH SCHMID

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INTRODUCTION

Recent achievements in the field of crystal plasticity receive considerable prominence in the present work. This is largely due to the extension of researches on plasticity to include metal crystals. Methods of producing such crystals and the determination of their orientation are therefore described in detail in Chapters III and IV. On the other hand, in view of the excellent treatises which already exist on the subjects of crystallography and crystal elasticity, the two introductory chapters under these headings have been severely limited. A description of the mechanisms of deformation, the geometry of which is expounded in Chapter V, is followed, in Chapter VI, by a fully detailed account of the application of these principles to metal crystals. Metal crystals have been accorded this preferential treatment on account of their usefulness in the experimental investigation of plasticity, and because much of our recent knowledge on the subject has been obtained with them. With the aid of the general principles which have been developed in this chapter the behaviour of ionic crystals is briefly treated in Chapter VII. The amount of experimental data collected in each of these chapters is certainly large, but, in view of the present unsatisfactory nature of the theories of crystal plasticity, this seemed unavoidable if the available material was to be adequately surveyed. Chapter VIII discusses a number of modern hypotheses, which it is hoped will soon be replaced by a single comprehensive theory. In the final chapter the knowledge acquired in our study of the single crystal is applied to elucidate the behaviour of polycrystalline material. Unfortunately the practical significance of this undertaking is still imperfectly realized by the technician.

The bracketed figures interspersed throughout the text refer to the list of publications at the end of the book, where the material has been classified by chapters to enable students who desire further information on certain aspects of the subject to find the appropriate references. It is, of course, inevitable that by this method the same work should sometimes appear under different numbers. Equations in the text bear first the number of the section in which they appear, and are then numbered consecutively.

Readers are referred to the following general treatises on crystal plasticity :

G. Sachs, "Plastic Deformation", *Handbuch der Experimental-physik*, Vol. 5/1, 1930.

A. Smekal, "Cohesion of Solids", *Handbuch der physikalischen und technischen Mechanik*, Vol. 4/2, 1931, and "Structure Sensitive Properties of Crystals", *Handbuch der Physik*, 2nd Edition, Vol. 24/2.

W. D. Kusnetzow, *The Physics of Solids*, Tomsk, 1932 (in Russian).

1. *Crystalline and Amorphous Solids*

Primarily, solids are commonly contrasted with liquids and gases, and are then divided into two fundamentally distinct groups: on the one hand, solids characterized by regular atomic arrangement (crystals), and on the other, amorphous materials of completely irregular structure. It has been found that a regular structure is by no means confined to those solids which, owing to their delimitation by plane surfaces, had already been recognized as crystals, but that it is, in fact, of very general occurrence throughout Nature.

The nature of the structure is of prime importance for the properties of a material, and the distinction between crystalline and amorphous states is revealed especially in plastic behaviour. With amorphous solids, deformation appears to occur by a mechanism of atomic migration under the influence of thermal movement, in the course of which the external forces merely bring about a preferential selection of those migrations which contribute to the relief of the imposed stresses. On the other hand, with crystals (which are the most important type of solids) the properties peculiar to a regular structure are also revealed in their plastic behaviour.

SOME FUNDAMENTALS OF CRYSTALLOGRAPHY

2. *Lattice Structure of Crystals*

Direct experimental proof of the lattice structure of crystals was made possible by von Laue's discovery of X-ray diffraction in crystals. To-day the regular arrangement of atoms (ions, molecules) in a three-dimensional "space-lattice" is considered characteristic of the crystalline state (Fig. 1). The shape and dimensions of the space-lattice are an expression of the forces prevailing between these smallest units of structure; they represent characteristic properties of the material, and they cannot be influenced by external agencies (*e.g.*, conditions of production).

According to present conceptions, plane surfaces are no longer regarded as a characteristic property of crystals. The "habit" of the crystals, *i.e.*, the external appearance implicit in the number and size of their faces, is thus shown to be an incidental and therefore unimportant feature. A sphere cut from a cube of rock salt, a grain of metal—these are as much crystals, exhibiting anisotropic properties, as the most perfect polyhedron of quartz.

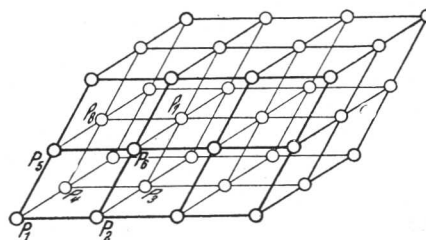


FIG. 1.—General Triclinic Space-lattice.

3. *Crystal Symmetry*

By contrast with the case of amorphous (vitreous) substances, different directions in crystalline material are, in general, no longer equivalent, owing to the lattice structure. Consequently the properties of crystals are usually different in different directions ("Anisotropy", cf. Section 58). It has been observed, however, that there are nearly always certain directions in which the geometrical and physical properties of the crystal are absolutely identical. It is the existence of this set of equivalent directions which imparts symmetry to the crystal. These observed symmetries must be regarded in terms of the space-lattice theory as lattice

symmetries. In any particular case the symmetry is represented by a so-called identity operation (or symmetry-group), a geometrical operation which, when applied to the lattice, brings equivalent directions and planes into self-coincidence.

The symmetry operations which are possible with a solid are rotations, reflexions and translations. A space-lattice, however, is severely limited in respect of rotations. An n -fold axis of rotation converts the lattice points, by rotation through angles $360^\circ/n$, into n equivalent positions, and so covers the plane at right angles to the axis with an unbroken series of points which form regular polygons of order n . But in accordance with our conception of the lattice, all points on a plane must also appear as the corners of a net of parallelograms. The only angles of rotation which fulfil both conditions are 360° (or 0°), 180° , 120° , 90° and 60° , corresponding to one-, two-, three-, four- and six-fold axes of rotation. These are, in fact, the only multiplicities observed in the symmetry axes of crystals.

All possible lattice configurations are obtained by combining the three elementary symmetry operations. The systematic development of these concepts, which is a purely mathematical operation, leads to the derivation of the 230 *space-groups* [Schönfliess (1891), Fedorow (1894)].

The lattice-translations leading to identity correspond to the distance between the smallest units of structure; they are of the order of 10^{-8} cm. and can be directly determined by measuring the crystal-lattice dimensions with the aid of X-rays. They represent a microscopic element of symmetry in contrast to the rotations and reflexions which can be observed macroscopically.

A combination of the macroscopic elements of symmetry alone results in a much more general division of observed forms into thirty-two *crystal classes* [Hessel (1830), Bravais (1849)].

Whereas previous classifications of the lattice shapes according to their symmetry properties proceeded along natural and obvious lines, further arrangement of the crystal classes into still larger groups was undertaken as a matter of convenience, the crystal classes being combined into *crystal systems*, based on the suitability of the co-ordinate system for defining the crystal. Fig. 2 shows the axes of co-ordinates corresponding to the six crystal systems.

The hexagonal-crystal system, which includes twelve crystal classes, is further subdivided into two groups. This is done either on the basis of the multiplicity of the principal axis in such a way that those crystal classes which have principal axes with only three-

fold (trigonal) symmetry are included in the trigonal subdivision; or alternatively, those crystal classes which are best described in

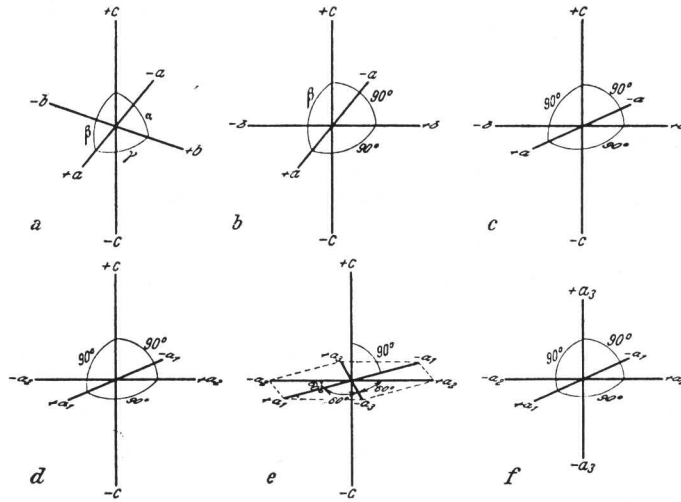


FIG. 2.—Axes of Co-ordinates for the Six Crystal Systems.
 (a) triclinic; (b) monoclinic; (c) orthorhombic; (d) tetragonal;
 (e) hexagonal; (f) cubic.

terms of rhombohedral co-ordinate axes (Fig. 3) are included in the rhombohedral subdivision.

Finally, mention must be made of two terms in frequent use; the *unit cell* and the *Bravais lattice*. The unit cell is that parallelepiped from which the whole crystal can be built up merely by parallel displacements (P_1P_2 to P_8 of Fig. 1). Bravais determined the number of different kinds of unit cell (described by specifying the unit-cell edges and angles) from which the space-lattice can be built up, by repetition of (parallel) translations in three dimensions. There are fourteen different translation groups, distributed in varying proportions among the individual crystal systems.

Although, in general, the lattices encountered are not simple “Bravais lattices”, they can always be regarded as resulting from the interpenetration of such simple translation lattices. Therefore, whereas space groups and crystal classes are classifications based on symmetry properties, the translation groups represent a system of

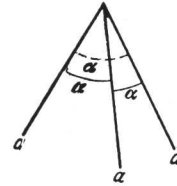


FIG. 3.—Rhombohedral Axis of Co-ordinates.

three-dimensional translations leading to the space-lattices; they afford no indication of the arrangement of the lattice units within the cell.

4. *The Crystallographic Representation of Planes and Directions*

The symmetry relationships of crystals have been described, and in discussing the crystal systems reference has been made to the systems of co-ordinate axes which can be conveniently used when defining crystals. We will now consider the method by which planes and directions (lattice planes and axes) can be described crystallographically.

Let us examine Fig. 4, in which a crystal plane $A_1B_1C_1$ is shown with the corresponding crystal axes XYZ . The plane makes intercepts $a_1b_1c_1$ on the three axes; all three intercepts are positive, since the plane lies entirely on the positive side of the origin. Assuming, as will be our practice in what follows, that the actual dimensions of the plane can be left out of account and that we are concerned solely with its orientation, it will suffice to indicate the ratios of the axial intercepts $a_1 : b_1 : c_1$. Another plane $A_2B_2C_2$ of the same crystal makes intercepts $a_2b_2c_2$, and the crystallographic *Law of Rational Indices*, which is based on considerable experimental data, states that there is invariably a relation of the type :

$$m_1a_1 : n_1b_1 : p_1c_1 = m_2a_2 : n_2b_2 : p_2c_2,$$

where m , n and p are simple whole numbers. This law becomes immediately intelligible if we consider the structure of the space-lattice.

The discussion can be greatly simplified and generalized if a prominent and specially important crystal plane is selected as reference or unit plane. Then the intercepts of all other planes are expressed as multiples of the corresponding intercepts of this unit plane, and, for instance, in specifying the space-lattice, the three translations of the lattice are selected as unit measurements on the respective axes.

If in our example the plane $A_1B_1C_1$ is chosen as unit plane, the figures $\frac{m_1}{m_2}$, $\frac{n_1}{n_2}$ and $\frac{p_1}{p_2}$ express the multiples of the unit distance cut off on the three axes by the plane $A_2B_2C_2$. The reciprocal values of these figures

$$h = \frac{m_2}{m_1}, k = \frac{n_2}{n_1} \text{ and } l = \frac{p_2}{p_1},$$

reduced to prime whole numbers, represent the *Miller indices* of the

plane $A_2B_2C_2$ or of other planes parallel to it.¹ The symbol of these crystallographically equivalent planes is written (hkl) . The unit plane, and the array of planes parallel to it, is designated (111) . A plane with a zero index, therefore, is parallel to the corresponding axis (intercepts it at infinity). Planes of co-ordinates passing through two axes are designated by two zero indices. When a plane makes equal intercepts along each of the three axes, the Miller indices are proportional to the direction-cosines of the plane normal.

Four axes are used in the hexagonal system, of which three are equivalent and lie in the basal plane. Since a plane is completely represented by the ratios of three figures, the four indices of the

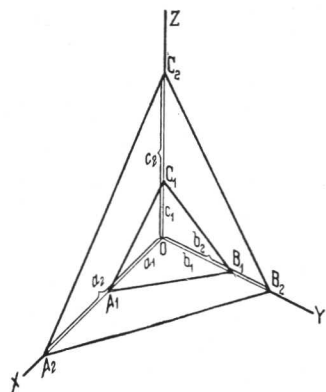


FIG. 4.—Diagram showing the Crystallographic Indexing of Faces.

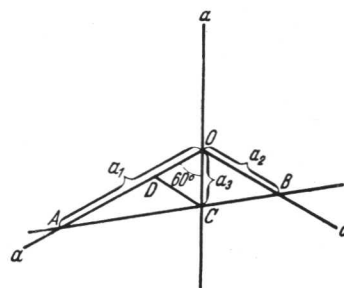


FIG. 5.—Intercepts on the Secondary Axes of the Hexagonal System.

hexagonal system are not independent. The relationship between indices, referred to the three two-fold secondary axes, can be easily deduced from Fig. 5, which shows the hexagonal basal plane. Let AB be the trace of the plane the indices of which are required, and a_1, a_2 and a_3 the intercepts on the digonal axes. Draw through C a parallel to OB and let it intercept the axis OA at D . We then obtain $OA : AD = OB : CD$, or $a_1 : (a_1 - a_3) = a_2 : a_3$, so that

$$a_3 = \frac{a_1 \times a_2}{a_1 + a_2}.$$

Transforming to the indices $h = \frac{1}{a_1}, k = \frac{1}{a_2}, i = \frac{1}{a_3}$, we have $\frac{1}{i} = \frac{1}{h + k}$.

¹ The reason for adopting these reciprocal intercepts is that they simplify the formulæ when calculating with crystallographic symbols.

Since the intercept on the a_3 axis is negative, we obtain finally $i = \overline{(h+k)}$,¹ i.e., the index applicable to the third axis is always equal to the negative sum of the first two.²

The crystallographic notation for *directions* is also based on the ratios of three figures. The line representing the direction passes through the origin of co-ordinates, and the co-ordinates are then determined for a given point lying on the line. These values are reduced to prime whole numbers, u , v and w , which are distinguished from the indices of a plane by being placed in square brackets $[uvw]$. One zero index indicates that the direction is parallel to one of the co-ordinate planes. The co-ordinate axes are expressed by the indices $[100]$, $[010]$ and $[001]$.³

The method of indexing planes and directions can now be described anew with the aid of the cubic crystal shown in Fig. 6 and the four-axial hexagonal crystal shown in Fig. 7. The cube faces in Fig. 6, which are parallel to the axial planes, have indices (100) , (010) and (001) . Of the four octahedral planes BDE has the indices (111) , while BDG making an intercept -1 on the Z axis has indices $(11\bar{1})$.

¹ The minus sign is always written *above* the corresponding index.

² In addition, the following relations hold between the hexagonal indices (hkl) and the rhombohedral indices (pqr) with first-order pyramidal plane $(10\bar{1}1)$ as rhombohedral plane :

$$p = 2h + k + l; \quad q = k - h + l; \quad r = -2k - h + l;$$

$$h = \frac{p - q}{3}; \quad k = \frac{q - r}{3}; \quad i = \overline{(h + k)}; \quad l = \frac{p + q + r}{3}.$$

³ In order to specify directions in the *four-axial* hexagonal system of co-ordinates, assume the direction Z through the origin and a given point P to be divided into four vector components :

$$Z = ua_1 + va_2 + ta_3 + wc.$$

This expression must naturally be identical with one which uses only three axes, for instance a_1 , a_2 , c .

$$Z = ma_1 + na_2 + wc.$$

For the secondary axes selected it will be true to say that the sum of their unit vectors, which form an equilateral triangle, disappears :

$$a_1 + a_2 + a_3 = 0.$$

If therefore in the above expression for Z , a_3 is replaced by $-(a_1 + a_2)$, then by comparing the coefficients we obtain :

$$u - t = m; \quad v - t = n,$$

u , v and t are still not clearly defined by these two equations (a vector can be resolved into three co-planar components in an infinite number of ways). The equation $u + v + t = 0$ is added as an arbitrary condition in the same way as when specifying the planes. It is now obvious that

$$u = \frac{2m - n}{3}; \quad v = -\frac{m - 2n}{3}; \quad t = -\frac{m + n}{3}.$$

At this stage, however, the indices no longer have any obvious geometrical significance.

4. The Crystallographic Representation of Planes and Directions 7

The two remaining octahedral planes are indicated by $(\bar{1}11)$ and $(1\bar{1}1)$. Of the six dodecahedral planes the plane $BDHF$ is specified by (110) , and the plane $ACGE$, which may be assumed to pass in a parallel direction through the point B or D , is marked $(1\bar{1}0)$ or $(\bar{1}10)$. The two sets of indices become identical on applying the reduction factor (-1) , which, as mentioned above, is always permissible.

Of the simple directions, reference should be made to $AB = [100]$ as one of the three edges of the cube, $AG = [111]$ as one of the four body diagonals, and $AF = [101]$ as one of the six face diagonals.

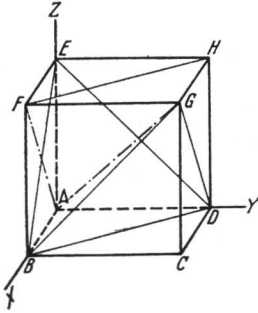


FIG. 6.—Method of Indexing the Faces and Directions of a Cubic Crystal.

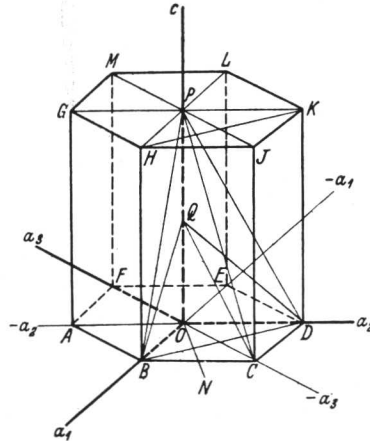


FIG. 7.—Method of Indexing the Faces and Directions of a Hexagonal Crystal.

The indices of the other crystallographically identical directions are obtained by transposing the indices (in cyclic order) and by using the negative sign. Other crystallographically important directions are those with indices $[112]$: these twelve identical directions connect a corner of the cube with a face centre on the opposite side.

The indices of some of the important planes of hexagonal crystals can now be easily stated (Fig. 7). The basal plane $ABCDEF$ is described by the symbol (0001) ; while the three prism planes type I ($BCJH$, $CDKJ$, $ABHG$) parallel to the digonal axes are indicated by $(10\bar{1}0)$, $(01\bar{1}0)$ and $(1\bar{1}00)$. The prism planes type II perpendicular to the digonal axes are indicated by $(11\bar{2}0)$ for $BDKH$, etc. The pyramidal planes type I which pass through the edges of the basal hexagon are indicated by $(10\bar{1}l)$, where l denotes

the order of the pyramid (*BCP* pyramid type I, order 1; *BCQ* pyramid type I, order 2). Pyramidal planes of type II correspond to the indices $(11\bar{2}l)$. (*BDP* pyramid type II, order 1; *BDQ* pyramid type II, order 2.) Among the directions to be noted are the hexagonal axis with the symbol $[0001]$, the digonal axes type I—*OB*, *OD* and *OF*—with indices $[\bar{2}110]$, $[1\bar{2}10]$ and $[11\bar{2}0]$, and finally one of the digonal axes type II, *ON*, with the indices $[10\bar{1}0]$.

5. Crystal Projection

Visual representation of the relationships between the angles of crystals, and simple methods for the performance of crystallographic

calculations, are provided by means of projections. There are two principal methods of projection, spherical and stereographic, both of which will now be briefly described.

In *spherical* projection a point on the crystal is assumed to be at the centre of an imaginary sphere. The crystal is then set up in such a way that a principal crystal axis emerges at the North and South poles. The projection of a *direction* is the point at which the line which has been drawn parallel to that direction and through the centre of the sphere meets the surface. The angle between two

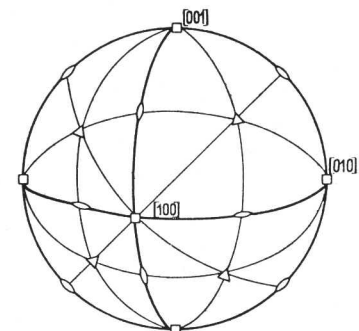


FIG. 8.—Spherical Projection of a Cubic Crystal.

□ () Δ : points at which the four-, three- and two-fold axes intersect the surface.

directions is therefore given by the angular distance between the representative points on the reference sphere. *Planes*, too, are represented by a point on the reference sphere, known as the “pole” of the plane, which is the point at which a plane normal drawn from the centre of the sphere intersects the surface. The angle between two planes is given by the distance between the two poles. The sum of all planes passing through one direction (a zone) is shown on the polar sphere by a great circle perpendicular to the common direction or zone axis. By representing the principal planes and directions in this way the symmetry of the crystals is impressed on the projection sphere (Fig. 8).

Crystallographic problems are solved by connecting the projection points (of planes and directions) on the polar sphere by great circles, the required angles being then calculated from convenient

triangles according to the formulæ of spherical trigonometry. The most commonly used formulæ are as follows, where a , b and c are the sides, and α , β and γ the angles of the triangle (Fig. 9) :

sine relation—

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}$$

cosine relation—

$$\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos \alpha \\ \cos \alpha &= -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a. \end{aligned}$$

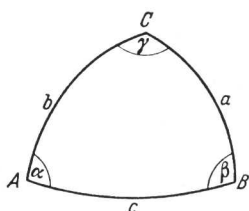


FIG. 9.—Spherical Triangle.

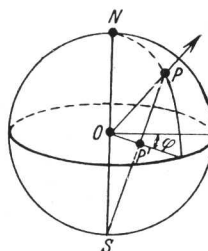


FIG. 10.—Spherical and Stereographic Projection of a Direction.

For the right-angled triangle ($\gamma = 90^\circ$)

$$\begin{aligned} \sin b &= \sin c \sin \beta \\ \sin a &= \sin c \sin \alpha \\ \cos c &= \cos a \cos b. \end{aligned}$$

In *stereographic projection* the polar sphere used in spherical projection is projected on to the equatorial plane, the northern hemisphere being viewed from the South pole, and the southern from the North pole (cf. Fig. 10).

The picture thus obtained is accurate in regard to angles, but not in regard to planes. Great circles of the polar sphere (crystallographic zones) project into circular arcs, and, when they pass through the point of projection, into diameters of the reference circle. Fig. 11, which should be compared with Fig. 8, contains the stereographic projection of a cubic crystal.

In this case the required angles are determined graphically with the aid of a ruled net. This consists of a number of equidistant meridians and parallel circles, with axis lying in the *equatorial plane* of the reference sphere (Fig. 12). In order to determine the angle

between two directions (A and B), the sphere (with ruled net) is rotated (see the arrow in Fig. 12a) until the points A and B are

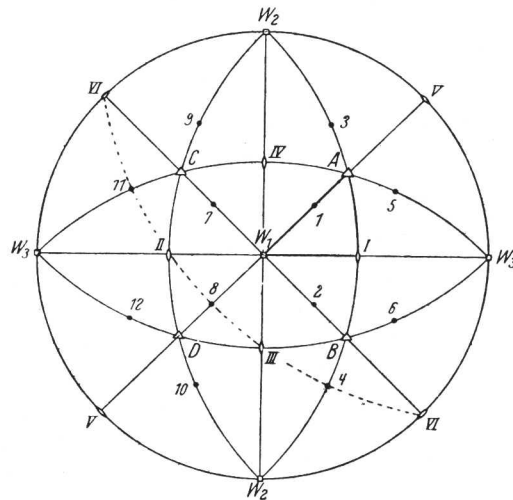


FIG. 11.—Stereographic Projection of a Cubic Crystal.

$W_1 \dots W_3$: cubic axes; $A \dots D$: body diagonals; $I \dots VI$: face diagonals; 1...12: $[112]$ directions.

connected by a great circle; the number of parallel circles between them gives the required angle. In stereographic representation, the

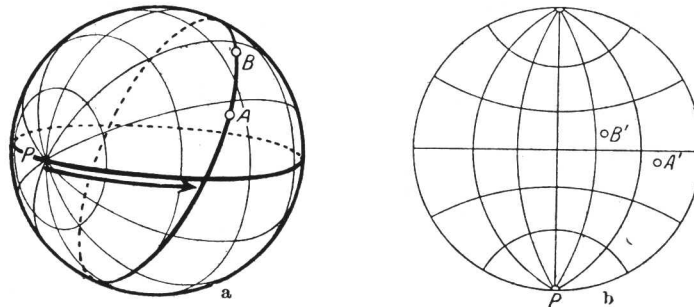


FIG. 12.—Ruled Net for Use in Stereographic Projection (according to Ewald).

Wulff net shown in Fig. 13 corresponds to the ruled sphere shown in Fig. 12b. If the Wulff net is placed beneath the transparent paper on which the projection has been traced and then turned

about its centre, this will correspond to rotation of the ruled sphere. Angles can be measured in this way to within about $\frac{1}{4}^\circ$, the accuracy attained depending on the distance between the circles on the net.

6. *Simple Crystallographic Theorems*

The use of crystallographic indices in calculations is illustrated below, by examples in common use.

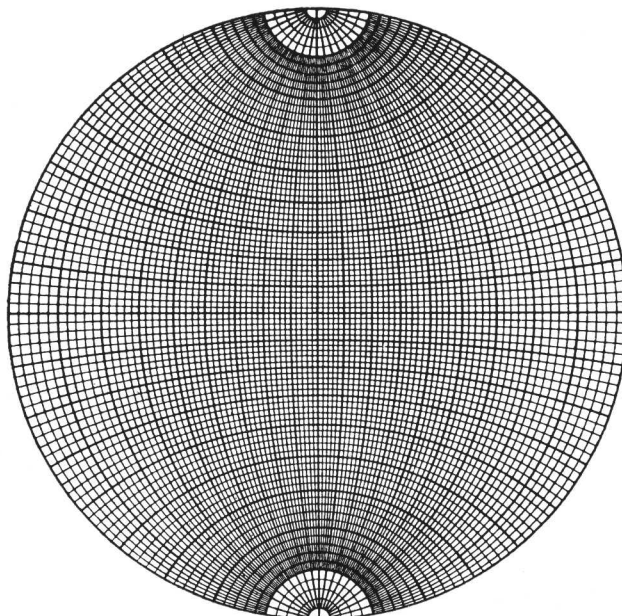


FIG. 13.—Wulff's Net.

(a) The direction $[uvw]$ lies in the plane (hkl) ; the plane (hkl) belongs to the zone $[uvw]$.

It follows from the analytical representation of planes and directions that the relation ¹ which must be satisfied for coincidence is $hu + kv + lw = 0$.

Thus, for example, the plane $(11\bar{2})$ belongs to the zone with axis in the direction $[111]$ and not to that with direction $[100]$.

(b) The intersection $[uvw]$ of two planes $(h_1k_1l_1)$ and $(h_2k_2l_2)$.

¹ For the four-figure indices of the hexagonal axis the analogous relation is

$$hu + kv + iw + lw = 0.$$

From $h_1u + k_1v + l_1w = 0$
 and $h_2u + k_2v + l_2w = 0$
 it follows that

$$\begin{aligned} u : v : w &= \left| \begin{array}{c} k_1l_1 \\ k_2l_2 \end{array} \right| : \left| \begin{array}{c} l_1h_1 \\ l_2h_2 \end{array} \right| : \left| \begin{array}{c} h_1k_1 \\ h_2k_2 \end{array} \right| \\ &= (k_1l_2 - l_1k_2) : (l_1h_2 - h_1l_2) : (h_1k_2 - k_1h_2). \end{aligned}$$

The planes (310) and (11 $\bar{1}$) intersect in the direction [$\bar{1}$ 32].

(c) Plane (hkl) through the two directions [$u_1v_1w_1$] and [$u_2v_2w_2$].

From the conditions of coincidence

$$\begin{aligned} u_1h + v_1k + w_1l &= 0 \\ \text{and } u_2h + v_2k + w_2l &= 0 \end{aligned}$$

it follows that $h : k : l = \left| \begin{array}{c} v_1w_1 \\ v_2w_2 \end{array} \right| : \left| \begin{array}{c} w_1u_1 \\ w_2u_2 \end{array} \right| : \left| \begin{array}{c} u_1v_1 \\ u_2v_2 \end{array} \right|$

[123] and [311] determine the plane (1 $\bar{8}$ 5).

(d) Transformation of indices to new axes.

It may sometimes be necessary to describe a crystal with reference to some axis other than the natural crystallographic axis. Examples of this are encountered in the indexing of layer-line diagrams. If, with reference to the original axes [100], [010] and [001], the new axes have indices [$u_1v_1w_1$], [$u_2v_2w_2$] and [$u_3v_3w_3$], then a plane with indices (hkl) in the original system has indices ($h'k'l'$) in the new system, and the transformation formulæ are :

$$\begin{aligned} h' &= u_1h + v_1k + w_1l \\ k' &= u_2h + v_2k + w_2l \\ l' &= u_3h + v_3k + w_3l. \end{aligned}$$

If a crystal is to be described, for instance, with the aid of the new axes [100], [010] and [112], then the following values are obtained for the new indices ($h'k'l'$) of a plane (hkl) :

$$h' = h, k' = k, l' = h + k + 2l.$$

(e) Spacing of lattice planes.

In view of the fundamental importance of the distance between equivalent lattice planes for the diffraction of X-rays in crystals, the general expression for this distance d may be given here. It is a function of the indices (hkl) of the plane: the coefficients are functions of the axial intercepts a, b, c and the axial angles α, β, γ .

The formula applicable to the general case (triclinic crystal) is

$$\frac{1}{d^2} = \frac{1}{V^2} \{g_{11}h^2 + g_{22}k^2 + g_{33}l^2 + 2g_{12}hk + 2g_{23}kl + 2g_{13}hl\}$$

$$\begin{aligned}
\text{where } g_{11} &= b^2c^2 \sin^2 \alpha & g_{12} &= abc^2(\cos \alpha \cdot \cos \beta - \cos \gamma) \\
g_{22} &= a^2c^2 \sin^2 \beta & g_{23} &= a^2bc(\cos \beta \cdot \cos \gamma - \cos \alpha) \\
g_{33} &= a^2b^2 \sin^2 \gamma & g_{13} &= ab^2c(\cos \gamma \cdot \cos \alpha - \cos \beta) \\
V^2 &= a^2b^2c^2 (1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2 \cos \alpha \cos \beta \cos \gamma)
\end{aligned}$$

V represents the volume of the elementary parallelepiped.

Special cases of higher symmetry :

$$\text{Orthorhombic } \frac{1}{d^2} = \left(\frac{h}{a}\right)^2 + \left(\frac{k}{b}\right)^2 + \left(\frac{l}{c}\right)^2$$

$$\text{Tetragonal } \frac{1}{d^2} = \frac{h^2 + k^2}{a^2} + \frac{l^2}{c^2}$$

$$\text{Cubic } \frac{1}{d^2} = \frac{h^2 + k^2 + l^2}{a^2}$$

$$\text{Hexagonal } \frac{1}{d^2} = \frac{4}{3} \cdot \frac{h^2 + k^2 + hk}{a^2} + \frac{l^2}{c^2} \quad (\text{valid for four-number indices with } i = h + k).$$

It may be said in general that the simpler the indices of a plane the greater is the interplanar spacing, and consequently the greater the number of lattice points per unit area of the plane (density of distribution). Thus, for example, for the cube face (100) of a cubic crystal, $d = a$, for the dodecahedral face (110),

$$d = \frac{a\sqrt{2}}{2},$$

while for the octahedral face (111),

$$d = \frac{a\sqrt{3}}{3}.$$

ELASTICITY OF CRYSTALS

7. Hooke's Law

If a solid body is subjected to mechanical stresses, elastic deformations will both precede and accompany plastic strain, *i.e.*, there will be changes in shape which disappear when the state of stress ceases. In this reversible process the deformation (which in any case is usually only very small) is determined solely by the prevailing stress, from which it can be calculated. The relationship between stress and strain is linear. This linear characteristic (Hooke's law), which is based on a wide experience, can now be deduced theoretically from Born's lattice theory, by assuming that the atoms in the crystal lattice are in positions of stable equilibrium relative to the lattice forces. The assumption is justified in so far as it has hitherto proved impossible to destroy, or even deform to any perceptible extent, a crystal by the application of infinitesimally small forces. It is assumed that the forces acting between the particles of the lattice are central forces; no assumptions are necessary regarding the law of inter-atomic forces itself when studying elastic behaviour. The distortion of a lattice has two components: the lattice is deformed as a whole; and, in addition, the simple lattices of which a crystal is generally composed can, as a whole, be displaced with reference to each other. This latter type of macroscopically invisible distortion is a peculiarity of the lattice structure of crystals.

The effect of external forces on a lattice is to displace the lattice points from equilibrium until the opposing forces set up by distortion re-establish equilibrium with the external forces. In order to calculate this behaviour we develop the energy density, whose derivatives with respect to the strain components are the stresses, in a power series of the strain components. The linear terms disappear owing to the assumption of the stability of the initial position: the components of a third and higher order are neglected. In this way the six equations of Hooke's generalized law are obtained:

$$\left. \begin{aligned} \sigma_x &= c_{11}\varepsilon_x + c_{12}\varepsilon_y + c_{13}\varepsilon_z + c_{14}\gamma_{yz} + c_{15}\gamma_{zx} + c_{16}\gamma_{xy} \\ \sigma_y &= c_{12}\varepsilon_x + c_{22}\varepsilon_y + c_{23}\varepsilon_z + c_{24}\gamma_{yz} + c_{25}\gamma_{zx} + c_{26}\gamma_{xy} \\ \sigma_z &= c_{13}\varepsilon_x + c_{23}\varepsilon_y + c_{33}\varepsilon_z + c_{34}\gamma_{yz} + c_{35}\gamma_{zx} + c_{36}\gamma_{xy} \\ \tau_{yz} &= c_{14}\varepsilon_x + c_{24}\varepsilon_y + c_{34}\varepsilon_z + c_{44}\gamma_{yz} + c_{45}\gamma_{zx} + c_{46}\gamma_{xy} \\ \tau_{zx} &= c_{15}\varepsilon_x + c_{25}\varepsilon_y + c_{35}\varepsilon_z + c_{45}\gamma_{yz} + c_{55}\gamma_{zx} + c_{56}\gamma_{xy} \\ \tau_{yx} &= c_{16}\varepsilon_x + c_{26}\varepsilon_y + c_{36}\varepsilon_z + c_{46}\gamma_{yz} + c_{56}\gamma_{zx} + c_{66}\gamma_{xy} \end{aligned} \right\} (7/1)$$