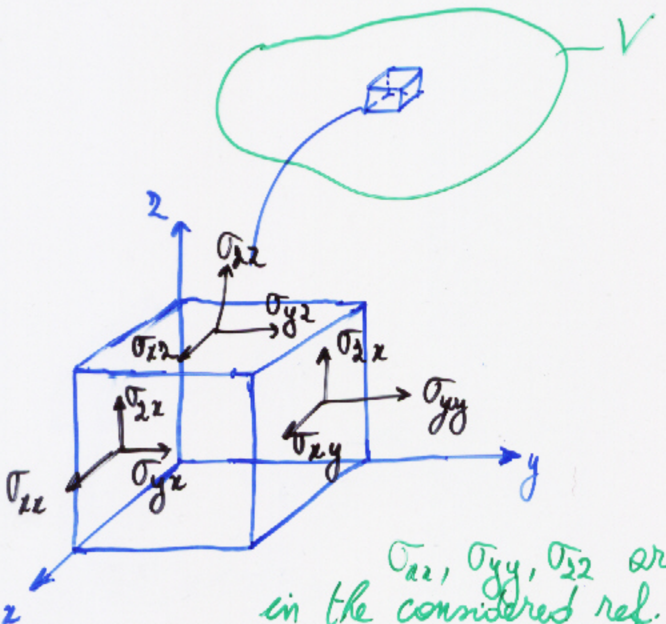
Parte 5

Description of stress at a point.

We consider the stress state of an infinitesimal small volume element of a 3D body.



$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ are normal stresses in the considered ref. system.

$\sigma_{xy}, \sigma_{yx}, \sigma_{xz}, \sigma_{zx}, \sigma_{yz}, \sigma_{zy}$ are shear stresses in the considered ref. system.

Schematic notation:

σ_{ij} are normal stresses for $i = j$

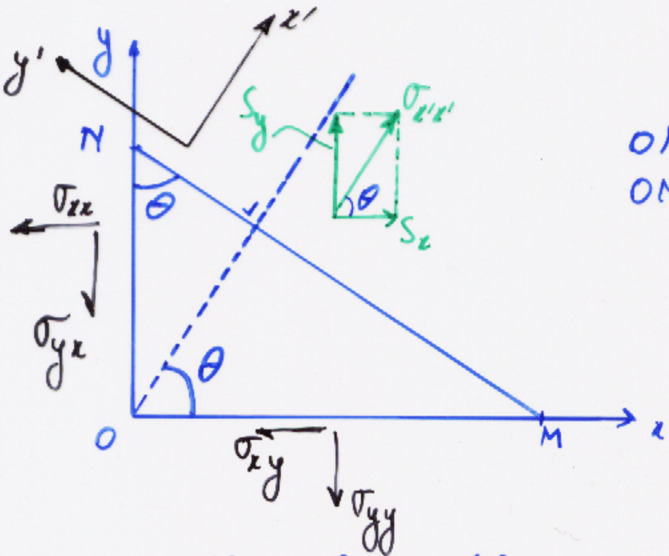
σ_{ij} are shear stresses for $i \neq j$

$i, j = 1, 2, 3$ and corresp.^{d} to x, y, z, resp.

Because of equilibrium of force momentum:

$$\sigma_{ij} = \sigma_{ji}$$

Stress state in two dimensions



$OM = A \sin\theta$
$OM = A \cos\theta$

A : surface of the oblique plane
SA = force on the oblique plane

→ x-comp^t of this force :

$$S_x A = \sigma_{xx} A \cos\theta + \sigma_{xy} A \sin\theta$$
$$S_x = \sigma_{xx} \cos\theta + \sigma_{xy} \sin\theta$$

→ y-comp^t of this force

$$S_y A = A\sigma_{yy} \sin\theta + A\sigma_{yx} A \cos\theta$$
$$S_y = \sigma_{yy} \sin\theta + \sigma_{yx} \cos\theta$$

Suppose we would define a new ref. system (x', y') as specified in the above figure

⇒ normal tension on x' plane is given by :

$$\sigma_{x'x'} = S_x \cos\theta + S_y \sin\theta$$

$$\Rightarrow \quad \sigma_{x'x'} = \sigma_{xx} \cos^2\theta + 2\sigma_{xy} \sin\theta + \cos\theta + \sigma_{yy} \sin^2\theta$$

And the shear tension in the x' plane is given by:

$$\sigma_{x'y'} = S_y \cos\theta - S_x \sin\theta$$

$$\sigma_{x'y'} = \sigma_{yy} \sin\theta\cos\theta + \sigma_{xy} \cos^2\theta - \sigma_{xx} \sin\theta\cos\theta - \sigma_{xy} \sin^2\theta$$

$$\sigma_{x'y'} = \sigma_{xy} (\cos^2\theta - \sin^2\theta) + (\sigma_{yy} - \sigma_{xx}) \sin\theta\cos\theta$$

And similarly, it can be proven that:

$$\sigma_{y'y'} = \sigma_{xx} \sin^2\theta + \sigma_{yy} \cos^2\theta - 2\sigma_{xy} \sin\theta\cos\theta$$

The above eq^s can be written in matrix notation:

$$\begin{bmatrix} \sigma_{x'x'} & \sigma_{x'y'} \\ \sigma_{x'y'} & \sigma_{y'y'} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

or briefly:

$$[\sigma_{ij}] = [a_{ik}] [\sigma_{kl}] [a_{jl}]^t$$

$$\sigma_{ij} = \sum_{k=1}^{2} \sum_{l=1}^{2} a_{ik} \, a_{jl} \, \sigma_{kl} \quad [1]$$

with $[a_{ij}]$ = the transformation matrix from the ref. system (x,y) to the ref. system (x',y').

	x	y
x'	a_{11}	a_{12}
y'	a_{21}	a_{22}

	x	y
x'	$\cos\theta$	$\sin\theta$
y'	$-\sin\theta$	$\cos\theta$

Einstein summation convention:
indices that appear twice are summed!

⇒ Eq. [1] can be written as:

$$\sigma'_{ij} = a_{ik}\, a_{jl}\, \sigma_{kl} \qquad [2]$$

The same expression also holds for 3D.

<u>Question</u>:
When is a physical quantity T_{ij} a tensor?

<u>Answer</u>:
A physical quantity T_{ij} is a tensor if and and only if it transforms as a tensor according to:

$$T'_{ij} = a_{ik}\, a_{jl}\, T_{kl}$$

with $[a_{ij}]$ = the tf. matrix.

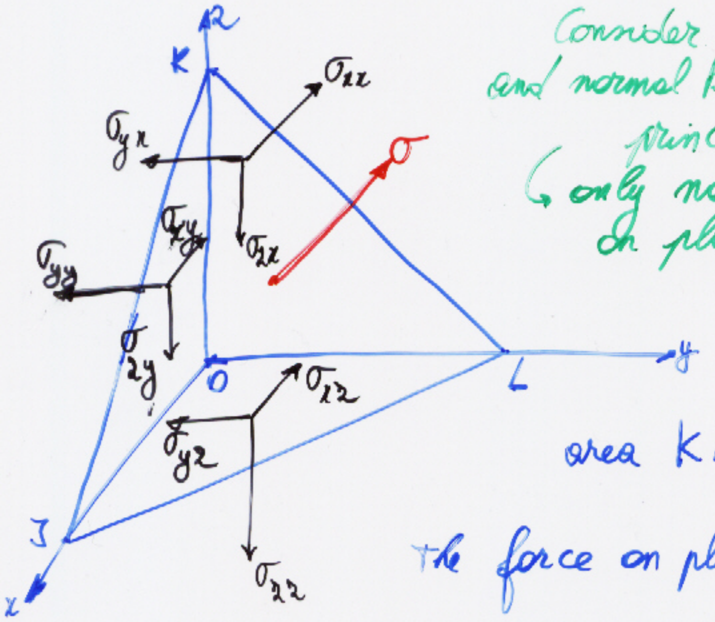
<u>Conclusion</u>:

By virtue of eq [2] the quantity σ_{ij} transforms as a tensor

⇒ The stress state σ_{ij} in an arbitrary point is a tensor property.

Principal Stresses.

Suppose: there exists a certain special ref. system for which the diagonal elements of the stress tensor disappear ($\sigma_{ij} = 0$ if $i \neq j$)
→ the principal ref. system.



Consider plane KLJ and normal to KLJ = principal direction
↳ only normal tension σ on plane KLJ

area KLJ = A

The force on plane KLJ = σA
= SA

⇒ decomposition of this force along (x, y, z)-axes:

$$S_x A = \sigma A l \ ; \quad S_y A = \sigma A m ; \quad S_z A = \sigma A n$$
$$S_x = \sigma l \ ; \quad S_y = \sigma m ; \quad S_z = \sigma n .$$

(with (l, m, n) = the direction cosines of $\perp$ KLJ).

The force balance in the x-direction:

$$\sigma l A = \sigma_{xx} A l + \sigma_{xy} A m + \sigma_{xz} n A$$
$$\Rightarrow (\sigma_{xx} - \sigma) l + \sigma_{xy} m + \sigma_{xz} n = 0$$

In analogy: force balances in y and z directions

$$\sigma_{yx}\, l + (\sigma_{yy} - \sigma) m + \sigma_{yz}\, n = 0$$

$$\sigma_{zx}\, l + \sigma_{zy}\, m + (\sigma_{zz} - \sigma) n = 0$$

In matrix notation:

$$\begin{bmatrix} (\sigma_{xx} - \sigma) & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & (\sigma_{yy} - \sigma) & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & (\sigma_{zz} - \sigma) \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \quad [1]$$

This is a system of 3 linear (homogeneous) equations in the unknowns (l, m, n).

$\Rightarrow$ there is a non-trivial solution

$\Updownarrow$

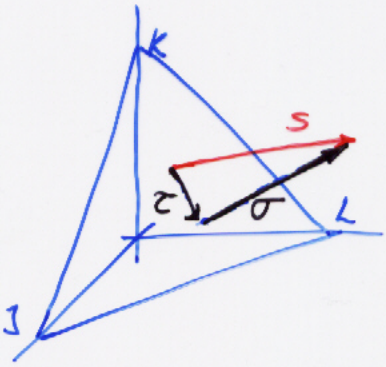
$$\begin{vmatrix} (\sigma_{xx} - \sigma) & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & (\sigma_{yy} - \sigma) & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & (\sigma_{zz} - \sigma) \end{vmatrix} = 0 \qquad [2]$$

$\rightarrow$ a 3rd order equation in σ

The three roots of this equation: $\sigma_1, \sigma_2, \sigma_3$
= the 3 principal stress.

In order to find the 3 principal directions (l_i, m_i, n_i) $\rightarrow$ substitute $\sigma_1, \sigma_2, \sigma_3$ in eq. [2] and find the corresponding solution for (l, m, n), respectively.

Suppose ⊥ KLJ ≠ principal direction.



KLJ = arbitrary plane

In this case: tension S on JKL ≠ normal tension

→ S can be decomposed in a normal comp[t] σ and a shear comp[t] τ.

$$S^2 = \sigma^2 + \tau^2 \quad [3]$$

The (x,y,z) comp[ts] of the force on KLJ =

$$S_x A,\ S_y A,\ S_z A$$

Force equilibrium in (x,y,z)-direction:

$$\begin{aligned} S_x A &= \sigma_{xx} A l + \sigma_{xy} A m + \sigma_{xz} A n \\ S_y A &= \sigma_{xy} A l + \sigma_{yy} A m + \sigma_{yz} A n \quad [4] \\ S_z A &= \sigma_{xz} A l + \sigma_{yz} A m + \sigma_{zz} A n \end{aligned}$$

The normal stress on KLJ is given by:

$$\sigma = S_x l + S_y m + S_z n$$

$$\sigma = [l\ m\ n] \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

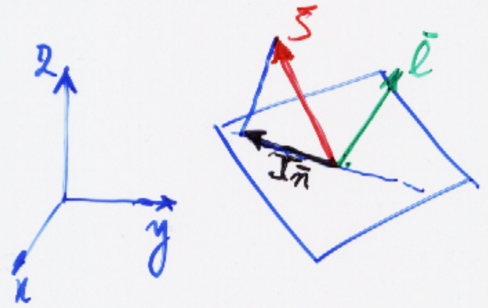
In full tensor notation:

$$\sigma = l_i \, \sigma_{ij} \, l_j \qquad [5].$$

According to eq. [4] the components S_i of a stress on an arbitrary plane with direction cosines (l, m, n) are given by:

$$S_i = \sigma_{ij} \, l_j .$$

The stress comp^t along an arbitrary direction n_i in the plane with direction co-sines l_j is given by:

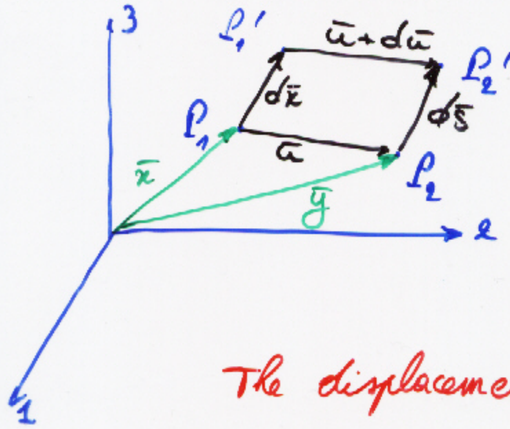


$$\boxed{\tau = n_i \, \sigma_{ij} \, l_j .}$$

Description of strain at a point

Displacement of points in a continuum
- rigid body translation
- rotation
- deformation

Deformation of a body
→ dilatation (change of volume)
→ distortion (change of shape)



The displacement vector $\bar{u}$

$$\bar{u} = \bar{y} - \bar{x}$$

$\bar{u}$ can be developed in a Taylor series around a point $\bar{r} = \bar{a}$

$$u_i = u_i(\bar{a}) + \frac{\partial u_i}{\partial r_j}(\bar{a})(r_j - a_j) + O(r_j - a_j)^2$$

$$\Rightarrow \bar{u} + d\bar{u} \; : \quad u_i + du_i = \frac{\partial u_i}{\partial x_j} dx_j + u_i \qquad (\blacksquare)$$

The vector $P_1 P_2'$ can be expressed in two different ways:

$$d\bar{x} + (\bar{u} + d\bar{u}) = \bar{u} + d\bar{s}$$

$$ds_i = dx_i + du_i$$

$$ds_i = dx_i + \frac{\partial u_i}{\partial x_j} dx_j$$

$$ds_i = \left(\delta_{ij} + \frac{\partial u_i}{\partial x_j} \right) dx_j$$

$e_{ij} = \dfrac{\partial u_i}{\partial x_j}$ = the displacement gradient tensor

The vector $d\bar{x}$ is transformed to $d\bar{s}$ in the following way:

$$\boxed{d\bar{s} = \left(\bar{\bar{I}} + \bar{\bar{e}} \right) d\bar{x}}$$

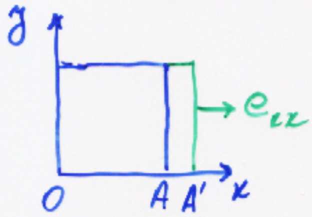
and from $(\blacksquare)$:

$$du_i = \frac{\partial u_i}{\partial x_j} dx_j$$

$$\boxed{du_i = e_{ij}\, dx_j}$$

$$e_{ii} = \frac{\partial u_i}{\partial x_i} \quad \text{(no summation)}$$

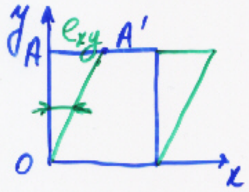
= the normal strains



$$e_{xx} = \frac{OA' - OA}{OA} = \frac{\Delta u_x}{\Delta x}$$

$$e_{ij} = \frac{\partial u_i}{\partial x_j} \quad i \neq j$$

= the shear strains



$$e_{xy} = \frac{AA'}{OA} = \frac{\Delta u_x}{\Delta y}$$

e_{ij} = gradient of displacement in i-direction of a planar surface $\perp$ j-direction

In general : $\bar{\bar{e}}$ = strain + rigid body rotation

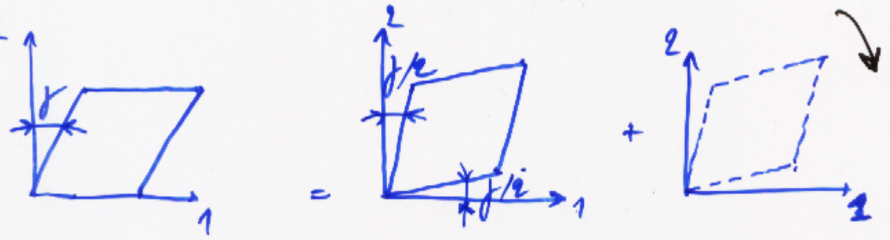
$$e_{ij} = \frac{1}{2}(e_{ij} + e_{ji}) + \frac{1}{2}(e_{ij} - e_{ji})$$

$\varepsilon_{ij} = \frac{1}{2}(e_{ij} + e_{ji})$ = strain tensor

$\omega_{ij} = \frac{1}{2}(e_{ij} - e_{ji})$ = rotation tensor

ε_{ij} = a symmetric tensor $\rightarrow \varepsilon_{ij} = \varepsilon_{ji}$

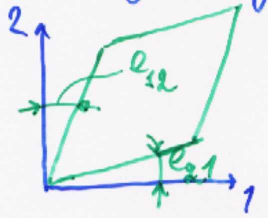
ω_{ij} = an anti-symmetric tensor $\rightarrow \omega_{ij} = -\omega_{ji}$



$$\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \gamma/2 \\ \gamma/2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \gamma/2 \\ -\gamma/2 & 0 \end{pmatrix}$$

total deformation = strain + rigid body rotation.

The engineering shear strain = angular change of a right angle



$$\gamma = e_{12} + e_{21} = \varepsilon_{12} + \varepsilon_{21}$$

$$\gamma = 2\,\varepsilon_{12} \qquad \boxed{\gamma_{ij} = 2\,\varepsilon_{ij}}$$

Question

Can the tensors ε_{ij} and ω_{ij} really be interpreted as a strain and a rigid body rotation?

Assume : $e'_{ij} = \varepsilon_{ij}$; $\bar{\bar{e}}' = \bar{\bar{\varepsilon}}$

$e''_{ij} = \omega_{ij}$; $\bar{\bar{e}}'' = \bar{\bar{\omega}}$

We know: $d\bar{s} = (\bar{\bar{I}} + \bar{\bar{e}})\, d\bar{x}$

$$\Rightarrow \quad d\bar{s} = (\bar{\bar{I}} + \bar{\bar{e}}'')(\bar{\bar{I}} + \bar{\bar{e}}')\, d\bar{x}$$

$$d\bar{s} = (\bar{\bar{I}} + \underbrace{\bar{\bar{e}}' + \bar{\bar{e}}'' + \bar{\bar{e}}''\bar{\bar{e}}'}_{\bar{\bar{e}}})\, d\bar{x}$$

$\bar{\bar{e}}$ only reduces to $\bar{\bar{e}} = \bar{\bar{\varepsilon}} + \bar{\bar{\omega}}$

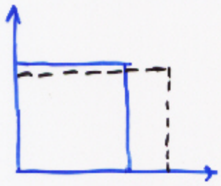
if and only if $\bar{\bar{e}}''\bar{\bar{e}}' \approx 0$

The 2nd order term $\bar{\bar{e}}''\bar{\bar{e}}'$ can only be neglected for <u>infinitesimal small strains</u>

Principal strains.

Question: Is there a reference system for which the shear strains $= 0$?

In such a reference system a vector $\bar{p} \parallel Ox_i$ is displaced parallel to its own direction



If $\bar{x} \parallel Ox_i \Rightarrow \bar{u} \parallel \bar{x}$

$$\bar{u} = \lambda \bar{x}$$
$$d\bar{u} = \lambda\, d\bar{x}$$

and $d\bar{u} = \bar{\bar{e}}\, d\bar{x}$

$$\Rightarrow \quad \bar{\bar{e}}\, d\bar{x} = \lambda\, d\bar{x}$$
$$e_{ij}\, dx_j = \lambda\, dx_j$$
$$(e_{ij} - \lambda \delta_{ij})\, dx_j = 0$$

There is a non-trivial solution if

$$| e_{ij} - \lambda \delta_{ij} | = 0$$

$$\begin{vmatrix} (e_{11} - \lambda) & e_{12} & e_{13} \\ e_{21} & (e_{22} - \lambda) & e_{23} \\ e_{31} & e_{32} & (e_{33} - \lambda) \end{vmatrix} = 0$$

This will produce a 3^{rd} order equation in λ. There are 3 real roots for sure if e_{ij} is symmetric, which is the case if we consider the strain comp^t of $\bar{\bar{e}}$

$\Rightarrow$ The 3rd order equation in λ :

$$\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0 \quad (*)$$

and $I_1 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_{ii}$

$$I_2 = \varepsilon_{11}\varepsilon_{22} + \varepsilon_{11}\varepsilon_{33} + \varepsilon_{22}\varepsilon_{33} - (\varepsilon_{12}^2 + \varepsilon_{23}^2 + \varepsilon_{32}^2)$$

$$I_3 = \varepsilon_{11}\varepsilon_{22}\varepsilon_{33} + 2\,\varepsilon_{21}\varepsilon_{31}\varepsilon_{23} - (\varepsilon_{11}\varepsilon_{23}^2 + \varepsilon_{22}\varepsilon_{13}^2 + \varepsilon_{33}\varepsilon_{12}^2)$$

I_1, I_2, I_3 are the 3 strain invariants

Assume: $\lambda_1, \lambda_2, \lambda_3$ are the 3 real roots of Eq. (*)

$\Rightarrow \quad \varepsilon_I = \lambda_1 \;;\; \varepsilon_{II} = \lambda_2 \;;\; \varepsilon_{III} = \lambda_3$

are the 3 principal strains ($\varepsilon_I > \varepsilon_{II} > \varepsilon_{III}$)

The 3 principal strain directions can be found by solving the following eq^n for $\lambda = \lambda_1, \lambda_2, \lambda_3$

$$\begin{bmatrix} (\varepsilon_{11}-\lambda) & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & (\varepsilon_{22}-\lambda) & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & (\varepsilon_{33}-\lambda) \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

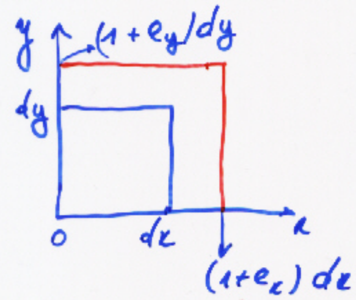
In analogy with principal shear stresses
$\rightarrow$ principal shear strains

$$\left\| \begin{aligned} \gamma_1 &= \varepsilon_2 - \varepsilon_3 \\ \gamma_2 &= \varepsilon_1 - \varepsilon_3 \\ \gamma_3 &= \varepsilon_1 - \varepsilon_2 \end{aligned} \right.$$

Deformation of a solid = volume change + shape change

Can strain tensor $\bar{\bar{\varepsilon}}$ be decomposed in a volume component + shape component ?

Δ = volume strain = change in volume per unit volume



$$\Delta = \frac{(1+\varepsilon_x)dx(1+\varepsilon_y)dy(1+\varepsilon_z)dz - dx\,dy\,dz}{dx\,dy\,dz}$$

$$\Delta = (1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z) - 1$$

$$\Rightarrow \quad \Delta \approx \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$\Delta = \varepsilon_{ii} = I_1$$ (1st invariant of the strain tensor)

Let's define $\varepsilon_m = \frac{\varepsilon_{ii}}{3} = \frac{\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}}{3} = \frac{\Delta}{3}$

ε_m = hydrostatic (spherical) strain component
= mean strain component

$$\varepsilon'_{ij} = \varepsilon_{ij} - \frac{\Delta}{3}\delta_{ij}$$

= the deviatoric strain tensor or strain deviator

→ describes the shape change

$$\varepsilon_{ij} = \varepsilon'_{ij} + \varepsilon_m \delta_{ij}$$

$$\boxed{\varepsilon_{ij} = \left(\varepsilon_{ij} - \frac{\Delta}{3}\delta_{ij}\right) + \left(\frac{\Delta}{3}\delta_{ij}\right)}$$

Plastic glide in single crystals.

Slip = primary deformation mechanism in most crystals

Properties:

- Shearing on crystallographic planes in crystallographic directions.
- Magnitude of shear = integral number of interatomic distances
- Slip = result of dislocation movement through the lattice
- Slip directions = crystal directions with the shortest repeat distance (e.g. $\langle 111 \rangle$ in BCC)
- Slip planes = dense packed planes.

Structure	Slip direction	Slip planes
fcc	$\langle 110 \rangle$	$\{111\}$
bcc	$\langle 111 \rangle$	$\{110\}, \{211\}, \{123\}$
hcp	$\langle 11\bar{2}0 \rangle, \langle 11\bar{2}3 \rangle$	$\{0001\}, \{1\bar{1}00\}, \{1\bar{1}01\}$, $\{1\bar{1}0\bar{1}\}$
dia. cub	$\langle 110 \rangle$	$\{111\}$
NaCl	$\langle 110 \rangle$	$\{110\}$
CsCl	$\langle 001 \rangle$	$\{100\}$
Fluorite	$\langle 110 \rangle$	$\{001\}, \{110\}, \{111\}$

Schmid's law

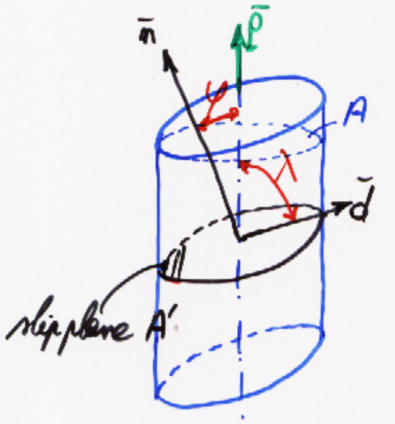
Slip $\Leftrightarrow$ shear stress τ_{nd} reaches a critical value τ_c

τ_c = the critical resolved shear stress

τ_{nd}: the resolved shear stress on a slip plane with normal n and along a slip direction d

Normally, slip occurs with equal ease in both directions (forward + backward slip).

$\Rightarrow$ slip $\Leftrightarrow$ $\tau_{nd} = \pm \tau_c$



Force $\bar{P}$ // x-direction

$$\tau_{nd} = \frac{P}{A'} \cos\lambda$$

and $A' = A/\cos\varphi$

$$\tau_{nd} = \frac{P}{A} \cos\lambda \cos\varphi$$

$$\tau_{nd} = \sigma_{xx} \cos\lambda \cos\varphi$$

$$\cos\varphi = l_{nx} \; ; \quad \cos\lambda = l_{dx}$$

⇒ Schmid's law:

$$\tau_{nd} = l_{nx}\, l_{dx}\, \sigma_{xx} = \pm \tau_c$$

$$\sigma_{xx} = \pm \frac{\tau_c}{l_{nx}\, l_{dx}} = \pm \frac{\tau_c}{m_x}$$

m_x = the Schmid factor for tension // x.

The generalized Schmid's law (tensor notation)

$$\tau_c = \bar{l}_d\, \bar{\bar{\sigma}}\, \bar{l}_n$$

$$\boxed{\tau_c = l_{ni}\, \sigma_{ij}\, l_{dj}}$$

$\bar{l}_n$ = unit vector // slip plane normal n

$\bar{l}_d$ = " " // slip direction d

<u>Example 1</u>

An aluminium crystal is tested in uni-axial tension // [112]. On which of the twelve fcc slip systems should slip occur? If the crystal yields when the applied tensile stress is 1.08 MPa, what is the value of τ_c?

Slip systems in fcc : {111} <110>

Calculate the resolved s.s. on each of the 12 slip systems:

e.g. $(111)[01\bar{1}]$ $\begin{cases} \bar{n} = \frac{1}{\sqrt{3}}[111] \\ \bar{d} = \frac{1}{\sqrt{2}}[01\bar{1}] \end{cases}$

$$\bar{x} = \frac{1}{\sqrt{6}}[11\bar{2}]$$

$$l_{nx} = \bar{n} \cdot \bar{x} = \frac{1}{\sqrt{3}}[111] \frac{1}{\sqrt{6}}[112]$$

$$l_{nx} = \frac{1}{\sqrt{18}}(1+1+2) = \frac{4}{3\sqrt{2}}$$

$$l_{dx} = \bar{d} \cdot \bar{x} = \frac{1}{\sqrt{2}}[01\bar{1}] \frac{1}{\sqrt{6}}[112]$$

$$l_{dx} = \frac{1}{\sqrt{12}}(1-2) = \frac{-1}{2\sqrt{3}}$$

$$m = l_{nx}\, l_{dx} = \frac{4}{3\sqrt{2}}\left(\frac{-1}{2\sqrt{3}}\right) = \frac{-2/3}{\sqrt{6}}$$

The slip system with the highest Schmid factor will be activated

(1) $(111)[01\bar{1}]$ $m = \frac{-2/3}{\sqrt{6}}$ (4) $(\bar{1}1\bar{1})[011]$ $m = -1/\sqrt{6}$

(2) $(111)[\bar{1}01]$ $m = \frac{2/3}{\sqrt{6}}$ (5) $(\bar{1}1\bar{1})[10\bar{1}]$ $m = \frac{1/3}{\sqrt{6}}$

(3) $(111)[1\bar{1}0]$ $m = 0$ (6) $(\bar{1}1\bar{1})[\bar{1}\bar{1}0]$ $m = \frac{2/3}{\sqrt{6}}$

(7) $(\bar{1}11)[01\bar{1}]$ $m = \frac{-1/3}{\sqrt{6}}$ (10) $(11\bar{1})[011]$ $m = 0$

(8) $(\bar{1}11)[101]$ $m = \frac{1}{\sqrt{6}}$ (11) $(11\bar{1})[1\bar{1}0]$ $m = 0$

(9) $(\bar{1}11)[\bar{1}\bar{1}0]$ $m = \frac{-2/3}{\sqrt{6}}$ (12) $(11\bar{1})[101]$ $m = 0$

The $(\bar{1}11)[101]$ and $(\bar{1}1\bar{1})[011]$ systems have the highest Schmid factors and should be active. The neg. value of m for slip syst. $(\bar{1}1\bar{1})[011]$ means that slip will occur in the $[0\bar{1}\bar{1}]$ direction or in the $+[011]$ direction on the $(1\bar{1}1)$ plane.

Slip occurs when $\sigma_{xx} = \pm\frac{\tau_c}{m_x}$

and $\sigma_{xx} = 1.08$ MPa ; $m_x = 1/\sqrt{6}$

$\Rightarrow |\tau_c| = m_x \sigma_{xx} = \frac{1.08}{\sqrt{6}} = 0.441$ MPa.

Example 2.

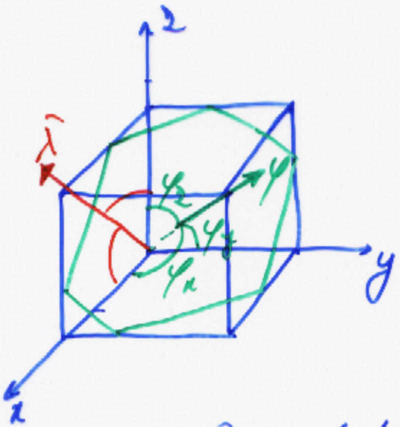
Consider the yielding of a single crystal with a single slip system. The angles that the x, y, z axes make with the slip plane normal $\bar{\varphi}$ and the slip direction $\bar{\lambda}$ are:

$\tau_c = 1.40$ MPa $\varphi_x = 45°$; $\varphi_y = 60°$; $\varphi_z = 60°$

$\lambda_x = 45°$; $\lambda_y = 120°$; $\lambda_z = 120°$

a. If the crystal is loaded under biaxial stress ($\sigma_x = \sigma_y \neq 0$), what is the value of σ_x at yielding?

b. If cryst. is loaded under simple shear $\tau_{yz} \neq 0$, calculate τ_{yz} at yielding?



$\varphi_x = 45°;\ \varphi_y = 60°;\ \varphi_z = 60°$

$\bar{\varphi} = \left(\frac{\sqrt{2}}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$\lambda_x = 45°;\ \lambda_y = 120°;\ \lambda_z = 120°$

$\bar{\lambda} = \left(\frac{\sqrt{2}}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$

$\tau_c = 1.40\ \text{MPa}$

a) bi-axial compression $\sigma_{xx} = \sigma_{yy} \neq 0$

$$\sigma_{zz} = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

The generalised Schmid law:

$$\tau_c = \bar{\varphi}\ \bar{\bar{\sigma}}\ \bar{\lambda} = 1.40$$

$$\frac{1}{2}\left[\sqrt{2}\ 1\ 1\right] \begin{pmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{xx} & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{2}\begin{bmatrix} \sqrt{2} \\ -1 \\ -1 \end{bmatrix} = 1.40$$

$$\frac{1}{4}\left[\sqrt{2}\ 1\ 1\right] \begin{bmatrix} \sigma_{xx}\sqrt{2} \\ -\sigma_{xx} \\ 0 \end{bmatrix} = 1.40$$

$$\frac{1}{4}\left(2\sigma_{xx} - \sigma_{xx}\right) = 1.40$$

$$\sigma_{xx} = -5.60\ \text{MPa}$$

Neg. value for $\sigma_{xx} = \sigma_{yy}$, because compression $\Rightarrow \tau_c = -1.40\ \text{MPa}$

b) Simple shear $\sigma_{y2} \neq 0$

$$\bar{\bar{\sigma}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma_{y2} \\ 0 & \sigma_{y2} & 0 \end{pmatrix}$$

The generalized Schmid law

$$\tau_c = \frac{1}{2}\begin{bmatrix} \sqrt{2} & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma_{y2} \\ 0 & \sigma_{y2} & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} \sqrt{2} \\ -1 \\ -1 \end{bmatrix} = 1.40$$

$$\frac{1}{4}\begin{bmatrix} \sqrt{2} & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -\sigma_{y2} \\ -\sigma_{y2} \end{bmatrix} = 1.40$$

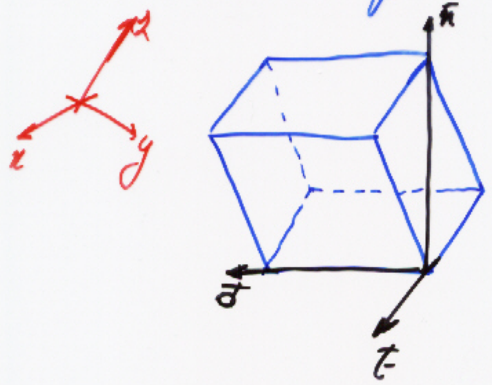
$$\frac{1}{4}\left(-\sigma_{y2} - \sigma_{y2}\right) = 1.40$$

$$-2\sigma_{y2} = 5.60$$

$$\underline{\sigma_{y2} = -2.8 \text{ MPa}}$$

Strains produced by slip.

We consider infinitesimal strain increments $d\varepsilon_{ij}$



$\bar{l}_x$: unit vector // $\bar{x}$

(l_{xd}, l_{xt}, l_{xn}) = comp^{ts} of $\bar{l}_x$ w.r.t. $(\bar{d}, \bar{t}, \bar{n})$

The coordinate tf. from $(\bar{d}, \bar{t}, \bar{n})$ axes to $(\bar{x}, \bar{y}, \bar{z})$ axes:

The strain comp^t $d\varepsilon_{xx}$ is given by:

$$d\varepsilon_{xx} = \bar{l}_x \; d\bar{\bar{\varepsilon}} \; \bar{l}_x$$

$$d\varepsilon_{xx} = \begin{bmatrix} l_{xd} & l_{xt} & l_{xn} \end{bmatrix} \begin{bmatrix} d\varepsilon_{dd} & d\varepsilon_{dt} & d\varepsilon_{dn} \\ d\varepsilon_{td} & d\varepsilon_{tt} & d\varepsilon_{tn} \\ d\varepsilon_{nd} & d\varepsilon_{nd} & d\varepsilon_{nn} \end{bmatrix} \begin{bmatrix} l_{xd} \\ l_{xt} \\ l_{xn} \end{bmatrix}$$

$$d\varepsilon_{xx} = d\varepsilon_{dd}\, l_{xd}^2 + d\varepsilon_{tt}\, l_{xt}^2 + d\varepsilon_{nn}\, l_{xn}^2 + 2\, d\varepsilon_{dt}\, l_{xd}\, l_{xt} + 2\, d\varepsilon_{tn}\, l_{xt}\, l_{xn} + 2\, d\varepsilon_{nd}\, l_{xn}\, l_{xd}$$

$2d\varepsilon_{ij} = d\gamma_{ij}$
$(i \neq j)$

$$d\varepsilon_{xx} = l_{xd}^2\, d\varepsilon_{dd} + l_{xt}^2\, d\varepsilon_{tt} + l_{xn}^2\, d\varepsilon_{nn} + l_{xd}\, l_{xt}\, \gamma_{dt} + l_{xt}\, l_{xn}\, \gamma_{tn} + l_{xn}\, l_{xd}\, \gamma_{nd}$$

Single slip on a single slip system in the $\vec{d}$ direction on the $\vec{n}$ plane

$\Rightarrow$ only shear term γ_{nd} in the above equation is $\neq 0$

$$\Rightarrow \quad d\varepsilon_{xx} = l_{xn}\, l_{xd}\, d\gamma_{nd}$$

In the Schmid notation : $l_{xn}\, l_{xd} = m$

$$\Rightarrow \quad d\varepsilon_{xx} = m\, d\gamma_{nd}$$

For other strain components :

$$d\varepsilon_{yy} = l_{yn}\, l_{yd}\, d\gamma_{nd}$$
$$d\varepsilon_{zz} = l_{zn}\, l_{zd}\, d\gamma_{nd}$$
$$d\gamma_{yz} = (l_{yn}\, l_{zd} + l_{yd}\, l_{zn})\, d\gamma_{nd}$$
$$d\gamma_{zx} = (l_{zn}\, l_{xd} + l_{zd}\, l_{xn})\, d\gamma_{nd}$$
$$d\gamma_{xy} = (l_{xn}\, l_{yd} + l_{xd}\, l_{yn})\, d\gamma_{nd}$$

Example 3

Consider the deformation of the crystal in example 2 and suppose the crystal is deformed until $\varepsilon_{yy} = -0.1$

What will be the values of the other strains?

Because ε_{yy} is small $\Rightarrow$ infinitesimal increment tf. laws may be applied.

$$\varepsilon_{yy} = l_{ym}\, l_{yd}\, \gamma = \left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)\gamma = -0.1$$

$$\Rightarrow \gamma = 0.4$$

$$\varepsilon_{xx} = l_{xm}\, l_{xd}\, \gamma = \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{\sqrt{2}}\right)(0.4) = 0.2$$

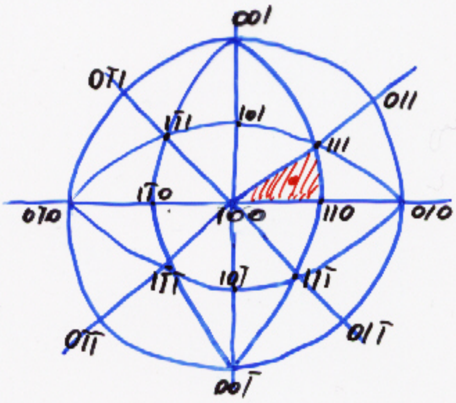
$$\varepsilon_{zz} = l_{zm}\, l_{zd}\, \gamma = \left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)(0.4) = -0.1$$

$$\gamma_{yz} = (l_{ym}\, l_{zd} + l_{yd}\, l_{zm})\gamma$$
$$= \left[\left(-\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)\right]0.4 = -0.20$$

etc.

Tensile deformation of fcc crystals.

Represent the tensile axis in the basic stereographic triangle



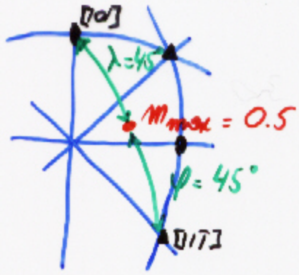
For all orientations of fcc tensile axes within the hatched stereographic triangle the Schmid factor for the $(11\bar{1})[101]$ slip system is higher than for any other system

↳ single slip on the primary slip system $(11\bar{1})[101]$

If the tensile axis (TA) is lying in any other stereographic Δ, the active slip system is found by simply noting the remote corners of the 3 adjacent triangles. The <111> in one is the slip plane normal (SPN) and the <110> in the other is the slip direction (SD).

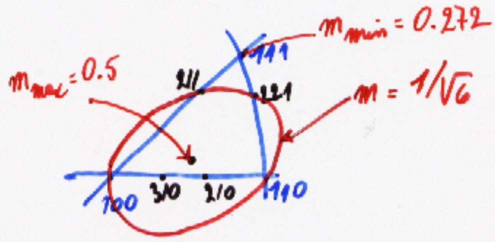
The maximum m-value : $m = 1/2$

if TA is on the great circle between σ and π and $\lambda = 45° = \varphi$



For TA located at [100], [110], [211] and [221]

$\rightarrow m = 1/\sqrt{6} = 0.408$



Lattice Rotation resulting from slip

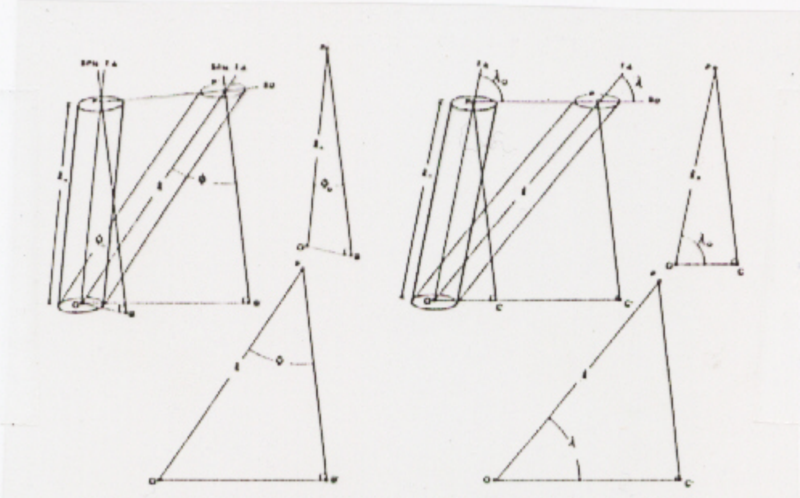


Fig. 2.7. Extension of a long thin crystal. Note that λ decreases and ϕ increases as the crystal is extended.

Crystallographic slip $\rightarrow$ lattice rotation

① slip does not change the distance between two slip planes

$$\overline{PB} = \overline{P'B'}$$

$$l_0 \cos\varphi_0 = l \cos\varphi$$

$$\Rightarrow \quad \frac{l}{l_0} = \frac{\cos\varphi_0}{\cos\varphi} = 1 + e$$

$$\cos\varphi = \frac{\cos\varphi_0}{1+e}$$

with $e = \frac{l - l_0}{l_0} = \frac{\Delta l}{l_0}$ = engineering strain.

② Slip does not change the distance between two parallel slip directions:

$$\rightarrow \quad \overline{PC} = \overline{P'C'}$$

$$l_0 \sin\lambda_0 = l \sin\lambda$$

$$\sin\lambda = \frac{\sin\lambda_0}{1+e}$$

with increasing strain e: $\lambda \downarrow$ and $\varphi \nearrow$.

The shear strain γ:

$$\gamma = \frac{\overline{PP'}}{\overline{PB}} = \frac{\overline{OC'} - \overline{OC}}{\overline{PB}}$$

$$\gamma = \frac{\overline{OC'}}{\overline{PB}} - \frac{\overline{OC}}{\overline{PB}} \quad \text{and } \overline{OC} = l_0 \cos\lambda_0$$

$$\overline{OC'} = l \cos\lambda$$

$$\overline{PB} = l_0 \cos\varphi_0 = l \cos\varphi$$

$$\Rightarrow \gamma = \frac{\cancel{l}\cos\lambda}{\cancel{l}\cos\varphi} - \frac{\cancel{l_0}\cos\lambda_0}{\cancel{l_0}\cos\varphi_0}$$

$$\boxed{\gamma = \frac{\cos\lambda}{\cos\varphi} - \frac{\cos\lambda_0}{\cos\varphi_0}}$$

<u>Question</u>: What is the rotation rate?

a) $l\cos\varphi = l_0 \cos\varphi_0 = \text{cte}$

$$\cos\varphi \, dl - l \sin\varphi \, d\varphi = 0$$

$$d\varphi = \cot g\, \varphi \, \frac{dl}{l} = d\varepsilon$$

$$\frac{d\varphi}{d\varepsilon} = \cot g\, \varphi$$

b) Similarly: $l \sin\lambda = l_0 \sin\lambda_0$

$$\Rightarrow \ldots \Rightarrow \frac{d\lambda}{d\varepsilon} = -\operatorname{tg}\lambda.$$

If TA is in the plane defined by SPN and SD $\Rightarrow \lambda + \varphi = 90°$

If not $\lambda + \varphi > 90°$

$$\Rightarrow \operatorname{cotg}\varphi \leq \operatorname{cotg}(90° - \lambda) = \operatorname{tg}\lambda$$

$$\Rightarrow \left|\frac{d\lambda}{d\varepsilon}\right| \geq \left|\frac{d\varphi}{d\varepsilon}\right|$$

→ the rate of rotation of the SD towards TA is larger than rotation rate SPN

For an fcc single crystal with TA in primary △ [100]-[110]-[111]: TA rotates on a great circle towards the primary slip direction [101]

↳ until it reaches [100]-[111] boundary
Along this boundary 2 slip systems are equally activated

→ $(11\bar{1})$ [101] → primary slip system
→ $(1\bar{1}1)$ [110] → conjugate " "

Net result: movement of TA along [100]-[111] boundary towards [211] pole
[211] is stable end orientation because it lies on the great circle through the 2 slip directions, and thus, 2 rotations cancel out each other.

Lattice rotation in compression (Taylor analysis)

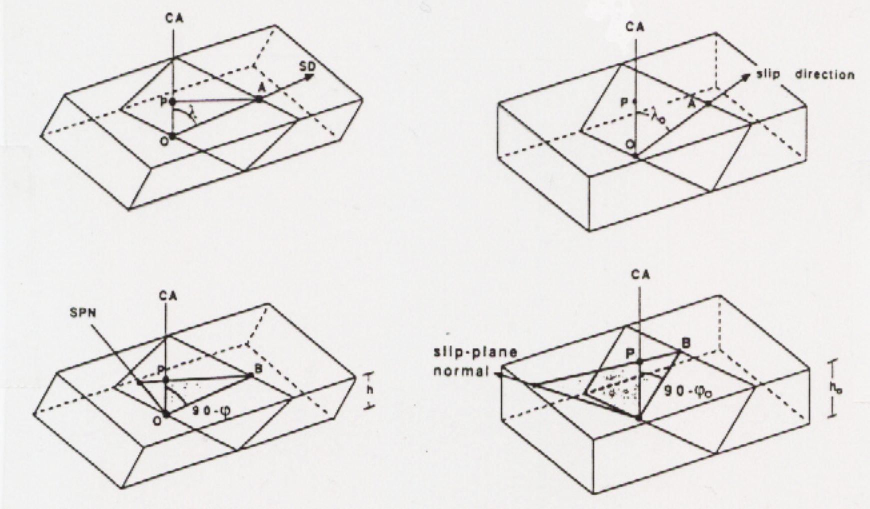


Fig. 2.12 Construction showing the lattice rotation of a thin flat crystal during compression. With compression λ increases and φ decreases.

Orientation change of a thin flat crystal during compression?

Compression axis CA $\perp$ compression plate

$\triangle$ OAP is a rectangular triangle

$h = \overline{OP}$ = thickness of sample

$$\rightarrow \quad \cos\lambda = \frac{\overline{OP}}{\overline{OA}} \quad \Rightarrow \quad \overline{OA} = \frac{h}{\cos\lambda}$$

$\overline{OA}$ = line in the slip plane

$\Rightarrow$ its length does not change

$$\Rightarrow \quad \overline{OA} = \frac{h_0}{\cos\lambda_0}$$

$$\Rightarrow \frac{h}{\cos\lambda} = \frac{h_0}{\cos\lambda_0}$$

$$\Rightarrow \frac{\cos\lambda}{\cos\lambda_0} = \frac{h}{h_0} = 1+e \quad \text{with } e = \frac{h-h_0}{h_0}$$

Plane through O : SPN $\cup$ CA

$\overline{OB}$ = intersection of this plane with SP.

$\triangle$ OPB is a rectangular triangle

$\varphi = \widehat{CA\ SPN}$

$$\Rightarrow \cos(90°-\varphi) = \frac{\overline{OP}}{\overline{OB}} \Rightarrow \overline{OB} = \frac{h}{\sin\varphi}$$

$\overline{OB}$ does not change during deformation

$$\Rightarrow \overline{OB} = \frac{h_0}{\sin\varphi_0}$$

and thus $\frac{h}{\sin\varphi} = \frac{h_0}{\sin\varphi_0}$

$$\frac{h}{h_0} = \frac{\sin\varphi}{\sin\varphi_0}$$

With compression $h/h_0 \downarrow \Rightarrow \varphi \downarrow$ and $\lambda \uparrow$

The relative rates of rotation :

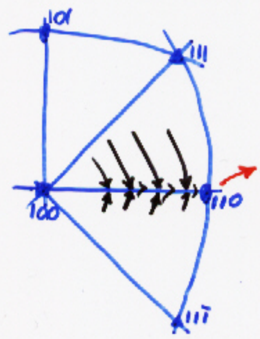
$$\frac{d\varphi}{-d\varepsilon} = -\operatorname{tg}\varphi$$

and $$\frac{d\lambda}{-d\varepsilon} = \operatorname{cotg}\lambda$$

with $-d\varepsilon = -\frac{dh}{h}$ = the compressive strain

$$\varphi + \lambda \geqslant 90° \Rightarrow \left|\frac{d\varphi}{d\varepsilon}\right| \geqslant \left|\frac{d\lambda}{d\varepsilon}\right|$$

→ rotation of CA towards SPN = primary rotation



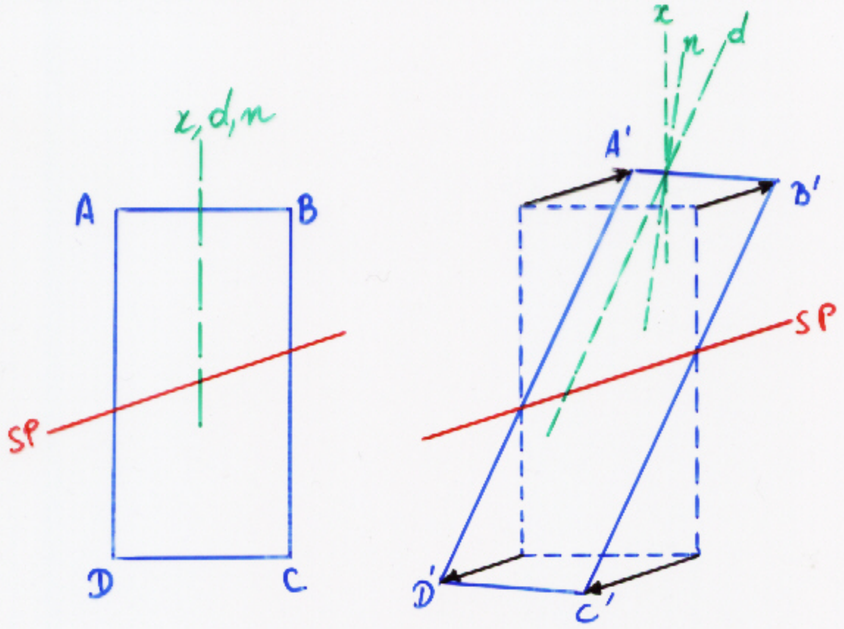
stable end orientation for CA

Discussion

Schmid analysis (tension)
vs. Taylor analysis (compression)

Schmid analysis : describes the rotation of the lattice relative to physical line in the material

Taylor analysis : describes the rotation of the lattice relative to a physical plane in the material.



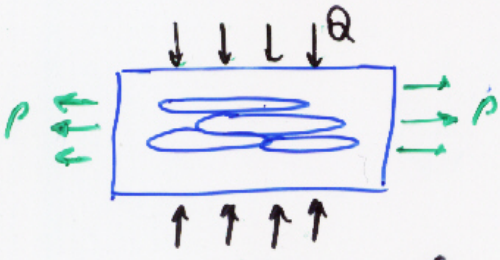
x = crystallographic direction

Before deformation : $x \parallel d$ and $x \parallel n$
After deformation : $x \nparallel d$ and $x \nparallel n$

In a tensile test: TA // longest dimension of the crystal

In a compression test: compression plates ⊥ CA

Lattice rotation of grains in polycrystals
→ texture development



Schmid Analysis

Q Taylor analysis.

Mathematical analysis of rotation.

Rotation of particles in a body with infinitesimal strain:

$$d\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

with $d\omega_{ij}$ = the clockwise rotation (in radians) about an axis $k \perp i, j$

u_i = shear displacement in the direction i of a plane $\perp j$.

The shear strain:

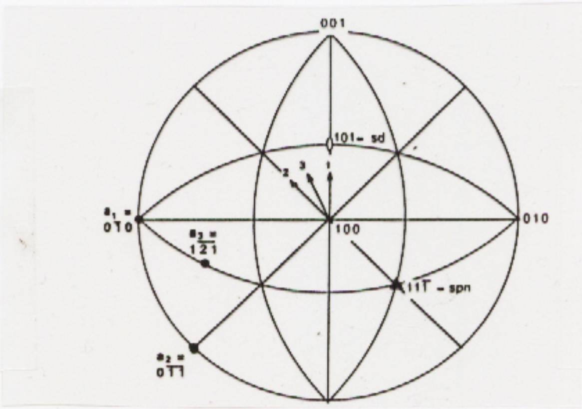
$$d\gamma_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$$

Suppose: i = slip direction
j = slip plane normal
$\partial u_j / \partial x_i = 0$

$$\Rightarrow \quad d\omega = \frac{1}{2} d\gamma$$

The rotation occurs about an axis $\perp$ SPN and SD

Prediction of the mathematical analysis are <u>intermediate</u> between the predictions from the Schmid and the Taylor analysis.



<u>Example</u>: tension in fcc crystal. TA // [100]

- Schmid analysis: TA rotates towards SPD [101]
  ↳ rotation axis = a_1 = $[0\bar{1}0]$
- Taylor analysis: TA rotates away from SPN $[11\bar{1}]$
  ↳ rotation axis = a_2 = $[0\bar{1}\bar{1}]$
- Mathematical analysis:
  ↳ rotation axis ⊥ SPN $[11\bar{1}]$ and SD [101]
    ↳ // $[11\bar{1}] \otimes [101] = [1\bar{2}1]$

↳ direction of the rotation predicted by the mathematical analysis are intermediate

Multiple Slip.

= slip occurring simultaneously on more than one slip system

e.g. simultaneous slip on the A plane in the a direction + on B plane in the b direction

$$|\tau_{Aa}| = |\tau_{Bb}| = \tau$$

with τ = CRSS

Furthermore, CRSS cannot be exceeded on any other system

$$\Rightarrow \quad |\tau_i| \leq \tau$$

External strains = sums of contributions from individual slip systems

→ For simultaneous slip on Aa, Bb, Cc, ... systems:

$$d\varepsilon_{xx} = l_{xA}\, l_{xa}\, d\gamma_{Aa} + l_{xB}\, l_{xb}\, d\gamma_{Bb} + \dots$$

$$d\varepsilon_{yy} = l_{yA}\, l_{ya}\, d\gamma_{Aa} + l_{yB}\, l_{yb}\, d\gamma_{Bb} + \dots$$

$$d\varepsilon_{zz} = \dots$$

$$d\gamma_{yz} = (l_{yA}\, l_{za} + l_{ya}\, l_{zA})\, d\gamma_{Aa} + (l_{yB}\, l_{zb} + l_{yb}\, l_{zB})\, d\gamma_{Bb} + \dots$$

Distribution of stress and strain in polycrystals

Theory of polycrystal plasticity :

- takes into consideration the deformation mechanisms of individual grains and crystallites
- relates phenomena on the microscopic scale to those on the macroscopic scale.

We must make some assumptions about stresses or strains of the individual grains

↓

assumptions on the distribution of stresses and strains in the polycrystal.

1] The Sachs hypothesis (1928)

All grains of the polycrystal are submitted to the same stress = the macroscopic stress.

The Schmid law applies to each individual grain of the polycrystalline aggregate :

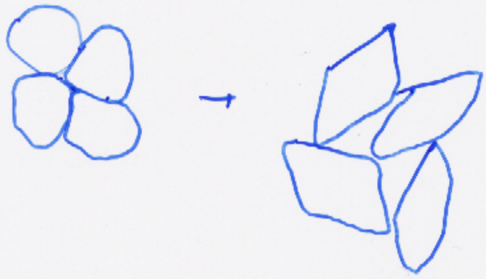
$$\tau^s = n_i^s \sigma_{ij} \nu_j^s$$

with ν_j^s = unit vector // slip plane normal
n_i^s = unit vector // slip direction.

- In general there is only one slip system activated per crystal orientation.

- Strain imposed on each crystallite = simple shear (along the slip plane, in the slip direction).

- The strain in one grain is NOT compatible with the strain in a neighbouring grain
→ holes exist between these grains.



- What about crystal rotation?

→ exactly the same situation as in single crystal plasticity.

→ an additional hypothesis is required on a boundary condition of the displacement:

e.g. physical line of the solid remains fixed in space (such as the tensile axis in a tension test) or, physical plane of the solid does not rotate (such as the compression plates in a compression test).

For complex cases → not very clear.

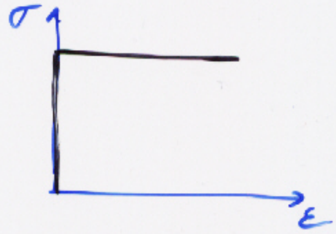
2] The Taylor hypothesis (1938).

All grains of the polycrystal are submitted to the same strain = the macroscopic strain.

→ provides the boundary condition of the displacement which is lacking in the Sachs theory → crystal rotations are uniquely determined in the Taylor theory.

→ The Taylor theory complies with strain compatibility, but there is no stress equilibrium between grains.

→ In the basic Taylor approach, we only consider the ideal plastic behaviour



* elastic strains are ignored
* no work hardening.

Lattice rotations in the Taylor model.

Purpose of the present analysis:

Calculate the orientation change of a particular grain caused by the prescribed (macroscopic) strain.

We consider to be known:

1] The macroscopic displacement gradient tensor $\bar{\bar{E}}$ (for small strain increments).

2] The orientation g of the grain <u>before</u> the strain occurs

3] The potential slip systems $\{hkl\}<uvw>$.

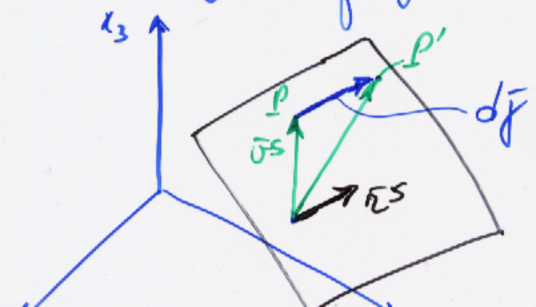
[1] and [2] are known within the sample reference system, whereas [3] is known in the crystal reference system.

Suppose: we consider an orientation g with a transformation matrix T

→ for each tensor the following transformation rule applies:

$$\bar{\bar{E}}^c = T\bar{\bar{E}}T^{-1} = T\bar{\bar{E}}T^t$$

with $\bar{\bar{E}}^c$ = the tensor property expressed in the crystal ref. system.



Suppose a shear $d\gamma^s$ on slip system s.

$d\gamma^s > 0$ because $d\bar{\gamma}^s // \bar{n}^s$

The displacement gradient tensor $\bar{\bar{E}}^s$ is given by:

$$d e_{ij}^s = n_i^s v_j^s d\gamma^s$$

with de_{ij}^s = the components of $\bar{\bar{E}}^s$ in the crystal reference system.

↳ describes the displacement associated with the slips $d\gamma^s$.

If the slip system is $(h_1\, h_2\, h_3)[k_1\, k_2\, k_3]$

$$\rightarrow \quad n_i = k_i / (k_i k_i)^{1/2}$$

$$v_i = h_i / (h_i h_i)^{1/2}$$

(for cubic systems)

Suppose: n = # of possible slip systems in a given grain.

↳ the displacement tensors $\bar{\bar{E}}^s$ must be calculated for these n slip systems

→ $\sum_s \bar{\bar{E}}^s$ = the total displacement gradient tensor associated with slips on <u>all</u> activated slip systems, taken together.

→ $\sum_s \bar{\bar{E}}^s$ + unknown rotation Ω^c of the crystal lattice

→ prescribed strain of the grain.

⚠ Strains are equal in all grains NOT the displacements

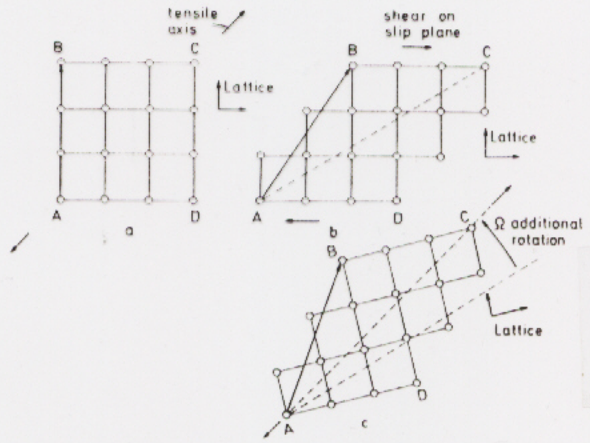


Fig. 5.5 a-b A shear γ on a slip plane does not cause the lattice to rotate, although a material vector may rotate (vector AB e.g.).
b-c An additional rotation - which also causes the crystal lattice to rotate - will bring the crystal in a position corresponding to the strain forced upon it : e.g. pure elongation in the direction AC.

Suppose $\bar{\bar{E}}^c$ = the crystallographic displacement tensor, expressed in the crystal ref. system.

Ω^c = the crystal lattice rotation, expressed in a non-rotating coordinate system which coincides with the crystal reference system before the crystal rotation.

$$\rightarrow \quad \bar{\bar{E}}^c = \Omega^c + \sum_{s=1}^{n} \bar{\bar{E}}^s$$

can be split in a symmetric and anti-symmetr. part:

$$\bar{\bar{E}}^c_s = \frac{1}{2}\left(\bar{\bar{E}}^c + \bar{\bar{E}}^{c\,t}\right) = \frac{1}{2}\sum_s \left(\bar{\bar{E}}^s + \bar{\bar{E}}^{s\,t}\right)$$

$$E^c_a = \frac{1}{2}\left(E^c - E^{ct}\right) = \Omega^c + \frac{1}{2}\sum_s \left(\bar{\bar{E}}^s - \bar{\bar{E}}^{s\,t}\right)$$

and the elements of $\bar{\bar{E}}_s^c$ should be equal to the elements $d\varepsilon_{ij}^c$ of the prescribed strain tensor.

$$d\varepsilon_{ij}^c = \frac{1}{2}\left(de_{ij}^c + de_{ji}^c\right)$$
$$= \frac{1}{2}\sum_s \left(n_i^s v_j^s + n_j^s v_i^s\right) d\gamma^s \quad [1]$$

with $d\varepsilon_{ij}$ are the elements of the prescribed strain tensor.

Eq. [1] is a linear system of eq^s:

- there are n unknowns $d\gamma^s$
- there are 6 eq^s corresponding to the 6 elements $d\varepsilon_{ij}^c$
  - ↳ but plastic deformation → NO volume change
    - ⇒ $d\varepsilon_{ii}^c = 0$
  - → there are only 5 independent eq^s.

In order to find a unique solution for this system
→ in the most general case : 5 active slip systems

If these $d\gamma^s$ values are known → the rotations Ω^c can be derived from :

$$d\omega_{ij} = \frac{1}{2}\left(de_{ij}^c - de_{ji}^c\right) \cdot \frac{1}{2}\sum_s \left(n_i v_j - n_j v_i\right) d\gamma_s \quad [2]$$

with $d\omega_{ij}$ the components of Ω^c

There are only 3 independent eq^s in [2]

In order to know the crystal rotation →
the Ω^c tensor must be transformed to
the sample reference system:

$$I + \Omega = T^t \left(I + \Omega^c \right) T \qquad [3]$$

with $I + \Omega$ = the rotation tensor that
describes the rotation of a material line
element $d\bar{x}$

Suppose : T' = the orientation matrix of the
new crystal, <u>after</u> the orientation change has
occurred :

$T' \rightarrow$

	RD	TD	ND
$(100)'$	a'_{11}	a'_{12}	a'_{13}
$(010)'$	a'_{21}	a_{22}	a_{23}
$(001)'$	a'_{31}	a'_{32}	a'_{33}

$T'^t \rightarrow$

	$(100)'$	$(010)'$	$(001)'$
RD	a'_{11}	a'_{21}	a'_{31}
TD	a'_{12}	a'_{22}	a'_{31}
ND	a'_{13}	a'_{23}	a'_{33}

$$\rightarrow \quad T'^t = (I + \Omega) T^t \qquad [4]$$

Eq. [4] describes the rotation of the crystal axes
<100> under the operation of the rotation tensor
$(I + \Omega)$

$$\text{Eq [4]} \rightarrow \quad T' = T (I + \Omega)^t$$

and given eq. [3]:

$$T' = T\,[T^t (I + \Omega^c) T]^t$$

$$T' = T\,T^t (I + \Omega^c)^t T$$

$$T' = (I + \Omega^c)^t\, T$$

and $(I + \Omega^c)^t = I - \Omega^c$
because Ω^c is an antisymmetric tensor.

$$\boxed{T' = (I - \Omega^c)\,T}$$

The new crystal orientation T' as a function of the old one T.

Taylor's second assumption.

The Taylor eq^s:

$$d\,\varepsilon^c_{ij} = \frac{1}{2} \sum_{s=1}^{n} (n_i^s v_j^s + n_j^s v_i^s)\, d\gamma^s \quad [1]$$

with n = number of slip systems.
Eq. [1] is a system of 5 linear independent eq^s with n unknowns.

If $n < 5 \Rightarrow$ there is NO solution for arbitrary strains $\bar{\bar{\varepsilon}}$

↳ the considered slip systems are inadequate to accommodate arbitrary strains.

In most cases $n > 5$ → Eq^s [1] will have solutions, but these solutions are not unique
→ there are ∞^{n-5} possible sets of solutions.

↳ a selection criterion is needed in order to select a meaningful solution out of all possible solutions.

→ Taylor's second assumption:

"The virtual work dW^* should be minimal for the meaningful solution".

For a given solution of eq. [1] dW^* is given by:

$$dW^* = \sum_{s=1}^{n} \tau_c^s \, |d\gamma^s|$$

Taylor's second assumption:

$$dW^* = \min.$$

If $\tau_c^s = cte \Rightarrow dW^* = \tau_c \sum_{s=1}^{n} |d\gamma^s|$

Taylor assumed that only 5 slip systems would be activated for the optimal solution

For fcc metals: 5 slip systems out of 12.

→ $C_{12}^5 = 792$ possible combinations.

For each comb. → eq. [1] is a set of 5 linear eq^s with 5 unknowns → unique solution.

Usually, there is more than one optimal solution
→ The Taylor ambiguity.

For the optimal solutions $dW^* = dW$,
and the minimum value is the same for all optimal solutions.

If $\tau_c = \tau_c^s = cte$ for all slip systems the same

$$\rightarrow \quad \frac{dW}{\tau_c} = \sum_{s=1}^{n} |d\gamma^s| = \min$$

Suppose: $d\Gamma = \sum_{s=1}^{n} |d\gamma^s|$

(for non-activated slip systems $d\gamma^s = 0$)

The Taylor factor M = the increase in total shear per unit macroscopic strain

$$M = \frac{d\Gamma}{d\bar{\varepsilon}}$$

e.g. the Von Mises equivalent strain:

$$d\bar{\varepsilon} = \frac{\sqrt{2}}{3}\left[(d\varepsilon_1 - d\varepsilon_2)^2 + (d\varepsilon_2 - d\varepsilon_3)^2 + (d\varepsilon_3 - d\varepsilon_1)^2\right]^{1/2}$$

with $(d\varepsilon_1, d\varepsilon_2, d\varepsilon_3)$ the principal strains.

When using another "macroscopic strain" than the Von Mises equivalent, the equivalent flow stress $\bar{\sigma}$ must be derived from:

$$\bar{\sigma} = \frac{dW}{d\bar{\varepsilon}}$$

Combining the above eq^s:

$$dW = \tau_c \, d\Gamma$$

$$dW = \tau_c M \, d\bar{\varepsilon}$$

and $\bar{\sigma} = M \tau_c$

The actual value of M depends on the conventional choice of $d\bar{\varepsilon}$.

The Bishop-Hill theory.

Aim of the theory:

. determine which slip systems are activated in each grain

. which stress is required to accomplish grain deformation.

- It is also assumed that each grain undergoes the same strain as the whole specimen.
- Taylor equations are <u>not</u> solved
- Taylor's 2nd assumption (minimization of virtual work) is substituted by the Maximum Work Principle (cfr. infra)

The maximum work principle uses the concept of the yield locus = representation of the yield condition in stress space.

In general: stress state is described by a symmetrical 3x3 tensor with 6 elements: σ_{ij}

Consider now a 6-dimensional space

→ any stress situation can be represented by a vector having the following elements:

$$\vec{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \sigma^1 \\ \sigma^2 \\ \sigma^3 \\ \sigma^4 \\ \sigma^5 \\ \sigma^6 \end{bmatrix}$$

The stress states which produce plastic flow are represented by a surface in 6D space, surrounding the origin = the yield locus.

Mathematical representation of the yield locus:

$$f(\sigma^k) = c$$

All physical stress states (whether elastic or plastic) must be represented by a point on or inside the yield locus:

$$f(\sigma^k) \leq c$$

Plastic yielding is not affected by the hydrostatic stress component σ'':

$$\sigma'' = \frac{1}{3}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)$$

→ the yield surface has a cylindrical shape
The generator of the cylinder is parallel with the following vector:

$$\vec{g} = [1\,1\,1\,0\,0\,0]^t$$

Also the strain tensor can be represented in a vector notation, by applying the following convent.

$$d\vec{\varepsilon} = \begin{bmatrix} d\varepsilon_{11} \\ d\varepsilon_{22} \\ d\varepsilon_{33} \\ 2d\varepsilon_{23} \\ 2d\varepsilon_{31} \\ 2d\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} d\varepsilon^1 \\ d\varepsilon^2 \\ d\varepsilon^3 \\ d\varepsilon^4 \\ d\varepsilon^5 \\ d\varepsilon^6 \end{bmatrix}$$

Is defined in such a way that the plastic work is given by the scalar product of the stress and strain "vectors":

$$dW = \sigma^i d\varepsilon^i$$

The Maximum Work Principle.

The work done by the actual stress in a prescribed (plastic) strain increment is greater than that done by any other stress not violating the yield criterion.

Consequences.

1. The yield surface must be plane or convex (i.e. concave on the side of the origin)

2. The strain increment vector must be normal to the yield surface and pointing outwards:

$$d\varepsilon^{k} = \frac{\partial f}{\partial \sigma^{k}}\, d\lambda \qquad (d\lambda \geqslant 0)$$

The Bishop-Hill analysis applies this concept to each individual crystallite of the polycrystal.

→ yield locus of the single crystal?

Yielding in a single crystal

⇕

absolute value of the resolved shear stress reaches the critical value on at least one of the slip systems:

$$n_i\, \sigma_{ij}\, v_j \leq \tau_c^{s} \qquad (1)$$

$$\text{or} \quad n_i\, \sigma_{ij}\, v_j \geqslant \tau_c^{s} \qquad (2)$$

Eq^{s} [1] and [2] define a polyhedron shaped subspace containing the allowed stress states.

↳ the single crystal yield locus.

If n slip systems → $2n$ plane sections.

Max work principle → only $2n$ directions are possible for the strain vector, except one the stress vector is pointing to an edge or a corner

Yield locus → cylindrical surface

Moreover: $d\bar{\varepsilon} \perp \vec{g}$ [11 1 000], because "constant volume" law:

$$d\varepsilon_1 + d\varepsilon_2 + d\varepsilon_3 = 0$$

$$d\vec{\varepsilon} \cdot \vec{g} = 0 \quad \Rightarrow \quad d\bar{\varepsilon} \perp \vec{g}$$

Consider now a <u>hyperplane</u>, containing the origin and perpendicular to $\vec{g}$

= 5D stress space

→ yield locus can be reduced to the intersection of this hyperplane and the original yield locus (→ the reduced yield locus).

→ the orthogonality rule also applies to the reduced yield locus, because $d\vec{\varepsilon} \perp \vec{g}$

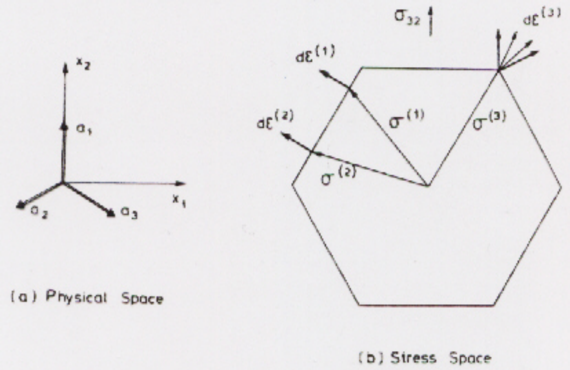


Fig. 5.7. (a) Definition of the coordinate system in the basal plane of a hexagonal crystal. a_1, a_2, a_3 are the potential slip directions.
(b) Section at $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{12} = 0$ through the yield locus of a hexagonal crystal yielding in basal slip. The stress states $\sigma^{(1)}$ and $\sigma^{(2)}$ will lead to the strain state $d\varepsilon^{(1)} = d\varepsilon^{(2)}$; any one of the strain increments $d\varepsilon^{(3)}$ will require the same stress $\sigma^{(3)}$ (Kocks (1970)).

Maximum Work Principles:

$\Rightarrow$ The stress vector corresponding to an arbitrary strain vector points to one of the corners of the yield locus.

$\Rightarrow$ When the strain vector is not arbitrary (e.g. it is perpendicular to an edge or to one of the 2n planes) $\rightarrow$ stress vector is allowed not to point to one of the corners

In the general case: the stress vector is always on one of the corners of the yield locus

The Bishop - Hill analysis:

1] Construct the single crystal yield locus.
2] Identify the corner stresses
3] Identify the activated slip systems associated with each corner stress.
(i.e. the hyperplanes intersecting at the corner)
4] Express the strain tensor in the crystal reference system and derive the strain "vector".
5] Calculate the scalar product of each of the corner stress vectors with the strain vector.
6] The correct corner stress is the one for which the product $dW = \vec{\sigma}\, d\vec{\varepsilon}$ attains a max. value.

$\rightarrow$ acting stress is known
+
active slip systems

⚠ However, the slips themselves remain unknown.

E.g. fcc metals: 12 slip systems {111} <110>
→ 56 corner stresses:
→ 24 activate 8 slip systems
→ 32 " 6 slip systems

Bishop-Hill and Taylor theories are exact equivalents.
→ two different mathematical formulations of the same optimization problem.

Bishop-Hill theory → individual slips on each slip system remain unknown.
→ lattice rotations cannot be determined, additional calculation steps are required.
↳ eg. Hansen (1975):
combination of B-H and Taylor.

e.g. • BH: 8 slip systems are active for a particular strain and a particular orient.

• Use these 8 slip systems to solve the Taylor equations
→ $C_8^5 = 56$ possible solutions (instead of 792 in original Taylor model!).

• Use Taylor's 2nd assumption to find a unique solution.